Variable horizon in a peridynamic body

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USNCCM12, Raleigh, NC, July 23, 2013
Outline

• Peridynamics background
  • States, horizon
• Rescaling a material model (at a point)
• Variable length scale (over a region)
• Partial stress
• Local-nonlocal coupling examples
Purpose of peridynamics

• To unify the mechanics of continuous and discontinuous media within a single, consistent set of equations.

Continuous body

Continuous body with a defect

Discrete particles

• Why do this? Develop a mathematical framework that help in modeling...
  • Discrete-to-continuum coupling
  • Cracking, including complex fracture patterns
  • Communication across length scales.

Figure 11.20 Pull-out: (a) schematic diagram; (b) fracture surface of ‘Silicom’ glass-ceramic reinforced with SiC fibres. (Courtesy H. S. Kim, P. S. Rogers and R. D. Rawlings.)
Peridynamics basics: The nature of internal forces

**Standard theory**
Stress tensor field
(assumes contact forces and smooth deformation)

\[ \rho \ddot{u}(x, t) = \nabla \cdot \sigma(x, t) + b(x, t) \]
Differentiation of contact forces

**Peridynamics**
Bond forces within small neighborhoods
(allow discontinuity)

\[ \rho \ddot{u}(x, t) = \int_{\mathcal{H}_x} f(q, x) \, dq + b(x, t) \]
Summation over bond forces

Horizon \( \delta \)

\( \mathcal{H}_x \) = family of \( x \)
Peridynamics basics: States

- A *state* is a mapping on bonds in a family.
- Notation:
  \[ A[x] \langle \xi \rangle \]
  where \( A \) is a state at a point \( x \) in a body, and \( \xi \) is a bond in the family of \( x \).
- Most of the states we deal with are vector-valued.

\[ \xi = q - x \]
Peridynamics basics:
Deformation state and force state

- The *deformation state* maps each bond to its deformed image.
  \[ Y[x] \langle q - x \rangle = y(q) - y(x) \]

- The *force state* maps each bond to bond force densities.
  \[ f(q, x) = T[x] \langle q - x \rangle - T[q] \langle x - q \rangle \]

- The *material model* at \( x \) maps deformation states to force states.
  \[ T[x] = \hat{T}(Y[x]) \quad T[q] = \hat{T}(Y[q]) \]
Peridynamics basics: Elastic materials

- A material is *elastic* if there is a scalar-valued *strain energy density function* 
  \[ \hat{W}(Y) \] such that
  \[ \hat{T} = \hat{W}_Y \]
- \[ \hat{W}_Y \] denotes the Frechet derivative with respect to the deformation state
  \[ \hat{W}(Y + dY) = \hat{W}(Y) + \hat{W}_Y \cdot dY \]
  for any small \( Y \).
- Here,
  \[ A \cdot B = \int_{H_x} A(\xi) \cdot B(\xi) \, d\xi \]

Differential work at \( x \) is
\[ d\hat{W} = T[x] \langle q - x \rangle \cdot dY[x] \langle q - x \rangle. \]
Scaling of a material model

- Suppose we have a material model $\hat{W}_\varepsilon$ for some given horizon $\varepsilon$.

- Try to find a material model with a new horizon $\delta$ such that $\hat{W}_\delta = \hat{W}_\varepsilon$ whenever the deformation is homogeneous.

- Define:
  
  $$r = \varepsilon / \delta \quad \xi_\varepsilon = r \xi_\delta \quad Y_\varepsilon \langle \xi_\varepsilon \rangle = r Y_\delta \langle \xi_\delta \rangle$$

- Our scaled material model is given by
  
  $$\hat{W}_\delta(Y_\delta) := \hat{W}_\varepsilon(Y_\varepsilon).$$

Material with horizon $\varepsilon$  

Material with horizon $\delta$
Scaling of the force state

- Now find the scaled force state \( \underline{T}_\delta \). We know

\[
dW = \underline{T}_\epsilon \cdot \underline{dY}_\epsilon = \underline{T}_\delta \cdot \underline{dY}_\delta.
\]

- Then

\[
\underline{T}_\epsilon \cdot \underline{dY}_\epsilon = \int_\epsilon \underline{T}_\epsilon \langle \xi_\epsilon \rangle \cdot \underline{dY}_\epsilon \langle \xi_\epsilon \rangle \, d\xi_\epsilon
\]

\[
= \int_\delta \underline{T}_\epsilon \langle \xi_\epsilon \rangle \cdot (r \underline{dY}_\delta \langle \xi_\delta \rangle) \, (r^D \, d\xi_\delta)
\]

\[
= r^{D+1} \underline{T}_\epsilon \cdot \underline{dY}_\delta
\]

where \( D \) is the number of dimensions.

- Hence the scaled force state is

\[
\underline{T}_\delta (\underline{Y}_\delta) = r^{D+1} \underline{T}_\epsilon (\underline{Y}_\epsilon), \quad r = \epsilon/\delta.
\]
Rescaling works fine if the horizon is independent of position

- Example: Uniform strain in a homogeneous bar.

\[ u = Hx \]

where \( H \) is a constant.

- If we set

\[ T_\delta \langle \xi_\delta \rangle = (\epsilon/\delta)^2 T_\epsilon \langle \xi_\epsilon \rangle \]

as derived above, we are assured the strain energy is independent of horizon.

- It follows that stress is also independent of horizon.
Variable horizon: the problem

• The scaling discussed above holds at any point $x$.

• We might anticipate that we can let horizon $\delta(x)$ vary with position and get the same result.

• Specifically, set

$$T[x] = \delta^{-2}(x) Z \left\langle \frac{\xi}{\delta(x)} \right\rangle$$

where $Z$ is a reference force state with horizon $\epsilon = 1$.

• Surprise! The uniform strain deformation is not in equilibrium even though it has uniform strain energy density.
Search for a different scaling

- Instead of
  \[ T[x] = \delta^{-2}(x) Z \left\langle \frac{\xi}{\delta(x)} \right\rangle \]

  search for pairs of functions \( \delta(x) \) and \( F(x) \) such that \( T \) given by

  \[ T[x] = F(x) Z \left\langle \frac{\xi}{\delta(x)} \right\rangle \]

  is equilibrated with no body force.

- This search has so far turned up only the following result:

  \[ \delta(x) = \delta_0 + ax, \quad F(x) = \delta^{-2}(x). \]

- This is interesting but not very practical since it doesn’t allow enough flexibility in modeling.
Interactions with variable horizon

- If the horizon is constant throughout the body, we write

$$\int_{\mathcal{H}_x} \left[ T[x] \langle q - x \rangle - T[q] \langle x - q \rangle \right] dq + b(x) = 0$$

where $\mathcal{H}_x$ is the family of $x$.

- If the horizon varies, we need to change the volume of integration to $S_x$, the “superfamily” of $x$ consisting of all the points that $x$ interacts with either through its own material model or the material model at the other points.

$$S_x = \mathcal{H}_x \cup \left\{ q \in \mathcal{B} \mid x \in \mathcal{H}_q \right\}$$
The peridynamic force density operator $L(x)$ involves the force state not only at $x$ but also the force states at all points within the horizon.

$$0 = L(x) + b, \quad L(x) = \int_{-\infty}^{\infty} \{T_{\delta(x)}[x][q - x] - T_{\delta(q)}[q][x - q]\} dq$$

so simply scaling the material model at $x$ is not sufficient.
Consider altering the equilibrium equation

- Use the 1D bar problem to define a “continuum patch test”.
- In a deformation of the form

\[ u(x) = u_0 + Hx \]

where \( u_0 \) and \( H \) are constants, and the material model is of the form

\[ T[x] \langle \xi \rangle = \delta^{-2}(x) Z(H) \left\langle \frac{\xi}{\delta(x)} \right\rangle \]

where \( Z \) is a state that depends on \( H \) only, require

\[ L(x) = 0 \]

for all \( x \) and for any prescribed \( \delta(x) \).
Peridynamic stress tensor

- The peridynamic stress tensor field* is defined by

\[ \nu(x) = \int_0^\infty \int_0^\infty \left\{ T[x-y](y+w) - T[x+y](y-w) \right\} \, dy \, dz. \]

- \( \nu \) allows the peridynamic force operator to be written in a form similar to the local theory:

\[ L(x) = \int_{-\infty}^{\infty} \left\{ T[x](q-x) - T[q](x-q) \right\} \, dq = \frac{d\nu}{dx}(x). \]

- \( \nu \) is the force per unit area carried by nonlocal interactions.

Partial stress field

- If $\delta$ and $T$ are constant, the peridynamic stress field simplifies to

$$\nu(x) = \int_{-\infty}^{\infty} \xi T[\xi] \langle \xi \rangle \, d\xi.$$ 

- Now use this expression to define a new field $\nu_0$ for any force state field:

$$\nu_0(x) := \int_{-\infty}^{\infty} \xi T[x] \langle \xi \rangle \, d\xi.$$ 

- In the patch test,

$$T[x] \langle \xi \rangle = \delta^{-2}(x) Z \left\langle \frac{\xi}{\delta(x)} \right\rangle \quad \implies \quad \nu_0(x) = \int_{-\infty}^{\infty} \xi Z \langle \xi \rangle \, d\xi$$ 

independent of $x$.

- $\nu_0$ is called the partial stress field.
Equilibrium equation based on the partial stress field

- The previous observation (that $\nu_0$ is independent of $x$ in the patch test) leads to the following proposed expression for internal force density:

$$L_0(x) = \frac{d\nu_0}{dx}(x)$$

where the equilibrium equation is

$$L_0(x) + b(x) = 0$$

and

$$\nu_0(x) = \int_{-\infty}^{\infty} \xi T[x, \langle \xi \rangle] d\xi$$

for any deformation, any $\delta(x)$, and any material model (possibly heterogeneous).

- Trivially, this model passes the patch test.
Concept for coupling method

- Idea: within a coupling region in which $\delta$ is changing, compute the internal force density from

$$L(x) = \frac{dv_0}{dx}(x), \quad v_0(x) = \int_{-\infty}^{\infty} \xi T[x](\xi) d\xi$$

instead of the full PD nonlocal integral.

- Here, $T[x](x)$ is determined from whatever the deformation happens to be near $x$.
  - $Z$ is no longer involved.
  - The material model has not changed from full PD, but the way of computing $L$ has.
Local-nonlocal coupling idea

Local region
\[ L(x) = \frac{dv_0}{dx} \]
\[ v_0(x) = \sigma(F(x)) \]

Transition region
\[ L(x) = \frac{dv_0}{dx} \]
\[ v_0(x) = \int \xi T[x] \langle \xi \rangle \, d\xi \]

Nonlocal region
\[ L(x) = \int \{ T[x] \langle \xi \rangle - T[x + \xi] \langle -\xi \rangle \} \, d\xi \]

Good old-fashioned local stress

Partial stress (PS)

Full peridynamic (PD)
Continuum patch test results

- Full PD shows artifacts, as expected.
- PS shows no artifacts, as promised.
Continuum patch test with coupling

- No artifacts with PD-PS coupling (this was hoped for but not guaranteed).

\[ u = 0 \]
\[ u = 0.02 \]
Pulse propagation test problem

- Does our coupling method work for dynamics as well as statics with variable horizon?

\[ \delta = 1 \text{ (nonlocal)} \]

\[ \delta = 0.01 \text{ (in effect local)} \]

Local-nonlocal transition region has width 1

Nonlocal

Local

Free boundary

Horizon

Force \( b \)

\( t \)
Pulse propagation test results

- Movies of strain field evolution

**Full PD everywhere**

**Coupled PD-PS**
Pulse propagation test results

- Strain field: no artifacts appear in the coupled model the local-nonlocal transition.
The partial stress approach may provide a means for local-nonlocal coupling within the continuum equations.

- Uses the underlying peridynamic material model but modifies the way internal force density is computed.
- Expected to work in 2D & 3D, linear & nonlinear.

PS is inconsistent from an energy minimization point of view.

- Not suitable for a full-blown theory of mechanics and thermodynamics (as full PD is).
- Not yet clear what implications this may have in practice.
- We still need to use full PD for crack progression.
Extra slides
Peridynamic vs. local equations

State notation: \( \text{State} \langle \text{bond} \rangle = \text{vector} \)

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<th>Standard theory</th>
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<td>( \mathbf{Y} \langle \mathbf{q} - \mathbf{x} \rangle = \mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{x}) )</td>
<td>( \mathbf{F}(\mathbf{x}) = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}(\mathbf{x}) )</td>
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<tr>
<td>Linear momentum balance</td>
<td>( \rho \ddot{\mathbf{y}}(\mathbf{x}) = \int_{\mathcal{H}} \left( \mathbf{t}(\mathbf{q}, \mathbf{x}) - \mathbf{t}(\mathbf{x}, \mathbf{q}) \right) dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x}) )</td>
<td>( \rho \ddot{\mathbf{y}}(\mathbf{x}) = \nabla \cdot \mathbf{\sigma}(\mathbf{x}) + \mathbf{b}(\mathbf{x}) )</td>
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<tr>
<td>Constitutive model</td>
<td>( \mathbf{t}(\mathbf{q}, \mathbf{x}) = \mathbf{T} \langle \mathbf{q} - \mathbf{x} \rangle ), ( \mathbf{T} = \hat{\mathbf{T}}(\mathbf{Y}) )</td>
<td>( \mathbf{\sigma} = \hat{\mathbf{\sigma}}(\mathbf{F}) )</td>
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<td>Angular momentum balance</td>
<td>( \int_{\mathcal{H}} \mathbf{Y} \langle \mathbf{q} - \mathbf{x} \rangle \times \mathbf{T} \langle \mathbf{q} - \mathbf{x} \rangle dV_{\mathbf{q}} = 0 )</td>
<td>( \mathbf{\sigma} = \mathbf{\sigma}^{T} )</td>
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<td>Elasticity</td>
<td>( \mathbf{T} = W_{\mathbf{Y}} ) (Fréchet derivative)</td>
<td>( \mathbf{\sigma} = W_{\mathbf{F}} ) (tensor gradient)</td>
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<td>First law</td>
<td>( \dot{\mathbf{\varepsilon}} = \mathbf{T} \cdot \ddot{\mathbf{Y}} + \mathbf{q} + \mathbf{r} )</td>
<td>( \dot{\mathbf{\varepsilon}} = \mathbf{\sigma} \cdot \dot{\mathbf{F}} + \mathbf{q} + \mathbf{r} )</td>
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\[ \mathbf{T} \cdot \ddot{\mathbf{Y}} := \int_{\mathcal{H}} \mathbf{T} \langle \xi \rangle \cdot \ddot{\mathbf{Y}} \langle \xi \rangle dV_{\xi} \]