Periodic Approximations Based on Sinc

Frank Stenger
School of Computing
University of Utah
sl Salt lake City, UT 84112 USA

Brandon Baker
Department of Electrical Engineering
University of Utah
Salt Lake City, UT 84112 USA

Carl Brewer
Department of Electrical Engineering
University of Utah
Salt Lake City, UT 84112 USA

Geoffrey Hunter
Department of Mathematics
University of Utah
Salt Lake City, UT 84112 USA

Sasha Kaputerko
Department of Geology and Geophysics
University of Utah
Salt Lake City, UT 84112 USA

Jason Shepherd
Scientific Computing and Imaging Institute
University of Utah
Salt Lake City, UT 84112 USA

Abstract

In this paper we derive some novel, "discrete wavelet" formulas for approximating a function f that is periodic with period T on $\mathbb{R} = \{x : -\infty < x < \infty\}$. At the outset we derive some families of approximations based the Whittaker Cardinal series expansion and

aliasing, including special cases of these, for interpolation at the points $\{k\,h\}_{-\infty}^{\infty}$ and $\{k\,h + h/2\}_{-\infty}^{\infty}$, combined with the cases of $T = 2\,N\,h$ and $T = (2\,N - 1)\,h$, where N is a positive integer. We also write down special cases of the formulas for the cases when f is either an even or an odd function on $\mathbb{R}$. The coefficients of each type of expansion are point evaluations of functions to be approximated, i.e., we differ from Fourier polynomial approximations in that no computations of the coefficients are required for our approximations. We then also derive some relations with polynomials in y via use of the transformation $y = \cos(2\,\pi\,x/T)$. We give some comparative examples of approximations of smooth periodic functions and discontinuous functions via both our periodic basis as well as with corresponding polynomial approximations.

1 Introduction and Summary

Fourier polynomial approximations are of course very important in applications.

2 Formula Derivations

We use two well known identities to derive our formulas, which are valid for all $z \in \mathbb{C}$:

$$\begin{aligned} \frac{\pi\,z}{\tan(\pi\,z)} &= \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{z-k}\,, \\ \frac{\pi\,z}{\sin(\pi\,z)} &= \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{z-k}\,. \end{aligned} \tag{2.1}$$

We shall assume throughout the paper that f is periodic on the real line $\mathbb{R}$, with period $T > 0$, i.e., that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. In addition, we shall assume that f takes on its mean value at all points on $\mathbb{R}$, which is especially important at points of discontinuity, i.e., that $\lim_{\delta\to 0}(f(x-\delta) - 2\,f(x) + f(x+\delta)) = 0$ for all $x \in \mathbb{R}$.

2.1 Formulas Based on the Cardinal Function

Let f have the above properties, and for a given number $a \in \mathbb{R}$ and $h > 0$, let us define the Cardinal series,

$$F(a, h, x) = \sum_{k \in \mathbb{Z}} f(k\,h - a\,h) \frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{\frac{\pi}{h}(x + a\,h - k\,h)}. \tag{2.2}$$

which approximates f on $\mathbb{R}$. We shall be primarily interested in two cases: $a = 0$ and $a = 1/2$.

Theorem 2.1 *Let f be periodic, with period T on $\mathbb{R}$.*

(i) If h is defined by $h = T/(2\,N)$, where N is a positive integer, then

$$F(a, h, x) = \sum_{k=0}^{2\,N-1} f(k\,h - ah)\, s(a, k, h, x), \tag{2.3}$$

where

$$s(a, k, h, x) = \frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{2\,N\ \tan\left\{\frac{\pi}{P}(x + a\,h - k\,h)\right\}}. \tag{2.4}$$

(ii) If h is defined by $h = T/(2\,N - 1)$, where N is a positive integer, then

$$F(b, h, x) = \sum_{k=0}^{2\,N-2} f(k\,h - b\,h)\, S(a, k, h, x) \tag{2.5}$$

where

$$S(a, k, h, x) = \frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{(2\,N - 1)\ \sin\left\{\frac{\pi}{T}(x + a\,h - k\,h)\right\}}. \tag{2.6}$$

Proof. *Part (i)*: Under the assumption that f has period T on $\mathbb{R}$, i.e., that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, and if a is an arbitrary real number, we have

$$
\begin{aligned}
F(a,h,x) &= \sum_{k\in\mathbb{Z}} f(k\,h - a\,h)\,\frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{\frac{\pi}{h}(x + b\,h - k\,h)} \\
&= \sum_{s\in\mathbb{Z}} \sum_{k=0}^{2\,N-1} f(k\,h - a\,h + 2\,s\,N\,h)\,\frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h - 2\,s\,N\,h)\right\}}{\frac{\pi}{h}(x + a\,h - k\,h - 2\,s\,N\,h)} \\
&= \sum_{k=-N}^{N-1} f(k\,h)\, s(a,k,h,x)\,,
\end{aligned}
\tag{2.7}
$$

and since $2\,N\,h = T$,

$$
s(a,k,h,x) = \frac{h}{\pi}\,\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}\,\sum_{s=-\infty}^{\infty} \frac{1}{x + a\,h - k\,h - s\,T}\,. \tag{2.8}
$$

Hence, using (2.1) (a), we get (2.4).

Part (ii): In this case, we have $(2\,N - 1)\,h = T$, so that

$$
\begin{aligned}
F(a,h,x) &= \sum_{k\in\mathbb{Z}} f(k\,h - a\,h)\,\frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{\frac{\pi}{h}(x + a\,h - k\,h)} \\
&= \sum_{s\in\mathbb{Z}} \sum_{k=0}^{2\,N-2} f(k\,h - a\,h + s\,T)\,\frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h - s\,T)\right\}}{\frac{\pi}{h}(x + a\,h - k\,h - s\,T)} \\
&= \sum_{k=0}^{2\,N-2} f(k\,h - a\,h)\, S(a,k,h,x)\,,
\end{aligned}
\tag{2.9}
$$

where

$$
S(a,k,h,x) = \frac{h}{\pi}\,\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}\,\sum_{s=-\infty}^{\infty} \frac{1}{x + a\,h - k\,h - s\,a}\,. \tag{2.10}
$$

Applying (2.1) (b) to this equation yields (2.6) .

■

Remark 2.2 The following identities are easily verified, for any given integer k ,

$$\begin{aligned} s(a,k,h,x) &= s(a,k+2\,N,h,x) \\ S(a,k,h,x) &= S(a,k+2\,N-1,h,x)\,. \end{aligned} \tag{2.11}$$

When these identities are combined with the above theorem they readily yield the following eight formulas of interpolation over the interval $[0,T]$ of even and odd periodic functions of period T defined on the real line $\mathbb{R}$.

(i) If $T = 2\,N\,h$, $a = 0$, and if f is an even function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$\begin{aligned} F(0,h,x) &= f(0)\,s(0,0,h,x) + f(Nh)\,s(0,N,h,x) \\ &\quad + \sum_{k=1}^{N-1} f(k\,h)\ \{s(0,k,h,x) + s(0,-k,h,x)\}\ . \end{aligned} \tag{2.12}$$

(ii) If $T = 2\,N\,h$, $a = 0$, and if f is an odd function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(0,h,x) = \sum_{k=1}^{N-1} f(k\,h)\ \{s(0,k,h,x) - s(0,-k,h,x)\}\ . \tag{2.13}$$

(iii) If $T = 2\,N\,h$, $a = 1/2$, and if f is an even function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(1/2,h,x) = \sum_{k=1}^{N} f(k\,h - h/2)\ \{s(1/2,k,h,x) + s(1/2,-k,h,x)\}\ . \tag{2.14}$$

(iv) If $T = 2\,N\,h$, $a = 1/2$, and if f is an odd function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(1/2,h,x) = \sum_{k=1}^{N} f(k\,h - h/2)\ \{s(1/2,k,h,x) - s(1/2,-k,h,x)\}\ . \tag{2.15}$$

(v) If $T = (2\,N-1)\,h$, $a = 0$, and if f is an even function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(0,h,x) = f(0)\,S(0,0,h,x) + \sum_{k=1}^{N-1} f(k\,h)\,(S(0,k,h,x) + S(0,-k,h,x))\,, \tag{2.16}$$

(vi) If $T = (2\,N-1)\,h,\ a = 0$, and if f is an odd function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(0,h,x) = \sum_{k=1}^{N-1} f(k\,h)\ \{S(0,k,h,x) - S(0,-k,h,x)\} \tag{2.17}$$

(vii) If $T = (2\,N-1)\,h,\ a = 1/2$, and if f is an even function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(1/2,h,x) = f(T/2)\,S(1/2,N,h,x) + \sum_{k=1}^{N-1} f(k\,h - h/2)\ \{S(1/2,k,h,x) + S(1/2,1-k,h,x)\} \tag{2.18}$$

(viii) If $T = (2\,N-1)\,h,\ a = 1/2$, and if f is an odd function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(1/2,h,x) = \sum_{k=1}^{N} f(k\,h-h/2)\ (S(1/2,k,h,x) - S(1/2,1-k,h,x)\} \tag{2.19}$$

3 Formulas Based on a Green's Function

An alternate class of trigonometric approximations is obtainable via use of the Green's function

$$G(x,y) = \frac{y}{\pi}\,\frac{1}{x^2+y^2} \tag{3.1}$$

for solution of Dirichlet problems in the upper half plane: $\{(x,y) \in \mathbb{R}^2\,,\ x \in \mathbb{R}\,,\ y > 0$. Although the coefficients of these bases are again function evaluations, this class does not interpolate, nor does it correspond to trigonometric polynomial approximation. However, it also does not exhibit Gibb's phenomena at discontinuities, i.e., it has properties similar to Fejér's method of trigonometric approximation.

It is well known that if f is continuous and uniformly bounded on $\mathbb{R}$, then

$$\lim_{y \to 0^+} \int_{\mathbb{R}} G(x - x', y)\, f(x')\, dx' = f(x) \tag{3.2}$$

at all points x of continuity of f.

The following approximation was studied in [S1, §5.8],

$$\mathcal{F}(\xi, y, h, x) \equiv \frac{h\, y}{\pi} \sum_{k \in \mathbb{Z}} \frac{f(\xi + k\, h)}{(x - \xi - k\, h)^2 + y^2}\,. \tag{3.3}$$

where it was shown, that if f is continuous and bounded on $\mathbb{R}$, and if β is an arbitrary fixed positive number in $(0,1)$, then $f_h(a, h^\beta, x) \to f(\xi + x)$ for all $x \in \mathbb{R}$.

Theorem 3.1 *If f is periodic, with period 2π on $\mathbb{R}$, and if $h = 2\pi/M$, with M a positive integer, then*

$$\mathcal{F}(\xi, y, h, x) = \sum_{k=0}^{M-1} f(\xi + k\, h)\, w(\xi, k, y, h, x)\,, \tag{3.4}$$

with

$$w(\xi, k, y, h, x) = \frac{\sinh(y)}{M\ \{\cosh(y) - \cos(x - \xi - k\, h)\}}\,. \tag{3.5}$$

Proof. We have

$$w(\xi, k, y, h, x) = \sum_{s \in \mathbb{Z}} G(x - \xi - kh - 2\pi\, s, y)\,, \tag{3.6}$$

and with $G(x, y)$ difined in (3.1).

Now, by applying partial fraction decomposition to the function $G(x - \xi - kh - 2\pi\, s, y)$, we find that

$$w(\xi,k,y,h,x) = \frac{i}{M\,\pi} \sum_{s\in\mathbb{Z}} \left\{ \frac{1}{(x-\xi-k\,h+i\,y)/(2\,\pi)-s} \right. \tag{3.7}$$
$$\left. - \frac{1}{(x-\xi-k\,h-i\,y)/(2\,\pi)-s} \right\},$$

and by using (1.3)-(a) and applying some trigonometric identities, we arrive at the expression (3.4) for w.

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4 Connections with Fourier Series and Numerical Integration

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It is convenient to take $T = 2\,\pi$. Two types of Dirichlet kernels are used regularly to prove the convergence of Fourier series, namely

$$\begin{aligned} D_s(N,\theta) &= \sum_{k=-N}^{N} e^{i\,k\,\theta} \\ &= \frac{\sin\{(N+1/2)\,\theta\}}{\sin(\theta/2)}, \end{aligned} \tag{4.1}$$

and also,

$$\begin{aligned} D_t(N,\theta) &= \frac{1}{2}\,e^{-i\,N\,\theta)} + \sum_{k=-N+1}^{N-1} e^{i\,k\,\theta} + \frac{1}{2}\,e^{i\,N\,\theta} \\ &= \frac{\sin\{N\,\theta\}}{\tan(\theta/2)}, \end{aligned} \tag{4.2}$$

Given F defined on $[-\pi\,,\pi]$, its Fourier series is given by

$$F(\theta) = \sum_{k\in\mathbb{Z}} c_k\,e^{i\,k\,\theta}, \tag{4.3}$$

with

$$c_k = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} F(\theta)\, e^{-i\,k\,\theta}\, d\theta\,. \tag{4.4}$$

It thus follows that

$$\frac{1}{2\,\pi} \int_{-\pi}^{\pi} F(\theta')\, D_s(\theta - \theta')\, d\theta' = \sum_{k=-N}^{N} c_k\, e^{i\,k\,\theta}\,, \tag{4.5}$$

and also, that

$$\frac{1}{2\,\pi} \int_{-\pi}^{\pi} F(\theta')\, D_t(\theta - \theta')\, d\theta' = \frac{1}{2} c_{-N}\, e^{-i\,N\,\theta} + \sum_{k=-N+1}^{N-1} c_k\, e^{i\,k\,\theta} + \frac{1}{2}\, c_N\, e^{i\,N\,\theta}\,. \tag{4.6}$$

The integrals in (4.5) and (4.6) are clearly continuous periodic wavelets.

Next, if $G(x, y)$ is defined as in (3.1), and if f is periodic on $\mathbb{R}$, with period $2\,\pi$, and proceeding as for the derivation of (3.6), we find that if f is periodic with period $2\,\pi$ on $\mathbb{R}$, we get

$$\begin{aligned} \mathcal{F}(y, x) &\equiv \int_{\mathbb{R}} G(x - x'\,, y)\, f(x')\, dx' \\ &= \frac{1}{2\,\pi} \int_{-\pi}^{\pi} \frac{\sinh(y)}{\cosh(y) - \cos(x - x')}\, f(x')\, dx'\,. \end{aligned} \tag{4.7}$$

The function $\mathcal{F}$ defined in this manner is thus also a continuous periodic wavelet.

5 Trapezoidal and Midordinate Integration

We should also mention two popular methods, the trapezoidal and midordinate rule of numerical integration over an interval $[0, T]$, with spacing $h = T/M$. First, the *trapezoidal rule*, which is given by

$$\int_0^T f(x)\, dx \approx \mathcal{T}_M(h, f)\,, \tag{5.1}$$

with

$$\mathcal{T}_M(h,f) = h\left\{\frac{1}{2}f(0) + \sum_{k=1}^{M-1} f(k\,h) + \frac{1}{2}f(T)\right\}\,, \tag{5.2}$$

and the *midordinate rule*,

$$\int_0^T f(x)\,dx \approx \mathcal{M}_M(h,f)\,, \tag{5.3}$$

with

$$\mathcal{M}_M(h,f) = h\sum_{k=1}^{M} f((k-1/2)\,h)\,. \tag{5.4}$$

Thus the formula in (2.3) may be obtained by applying the trapezoidal rule (5.1)–(5.2) to the integral (4.6) involving the kernel D_t, while (2.5) can be derived by applying the midordinate rule to the integral (4.5). Indeed, Gabdulhaev obtained (2.3) for the interval $[-\pi\,,\pi]$ in just this manner (see [S1, Theorem 2.2.6]).

6 Connection with Polynomial Approximation

Suppose that we wish to approximate a given function $F(y)$ on a finite interval (c,d) of the real line $\mathbb{R}$. Then, setting $y = (1/2)(c+d) + (1/2(d-c)\cos(x)$, we get a new function, $G(x) = F(y) = F((1/2)(c+d)+(1/2)(d-c)\cos(x))$, which is a periodic function on all of $\mathbb{R}$, with period 2π, and moreover, $G(x)$ is an even function of x. We can thus approximate G on the interval $[0,\pi]$ via use of either (2.7) or (2.13). Since G is representable via a rapidly convergent trigonometric cosine series expansion, as can be seen from either (3.5) or (3.6), and since $\cos(m\,x) = T_m(w)$, with $w = \cos(x)$, and with $T_M(w)$ denoting the Chebyshev polynomial, it follows that the corresponding function $F(y)$ is now approximated via a Chebyshev polynomial expansion in the variable

$$w = \frac{y-(1/2)(c+d)}{(1/2)(d-c)}\,. \tag{6.1}$$

Moreover, this approximation is a rapidly convergent function of N if F is analytic in an open region containing the interval $[c,d]$. However, we no longer have interpolation at a set of equi-spaced points, but rather at the points [1]

$$y_j = (1/2)(c+d) + (1/2)(d-c)\,\cos(x_j)\,, \tag{6.2}$$

where $x_j = \pi\, j/N$, $j = 0\,,\ 1\,,\ \ldots\,,\ N$ for the case of (2.7), and with $x_j = \pi(j-1/2)/N$, $j = 1\,,\ 2\,,\ \ldots\,,\ N$ for the case of (2.13) .

We may thus note the relations: *Chebyshev polynomial approximation subset Trigonometric polynomial interpolation* $\subset$ *Sinc series.*

7 Examples

We illustrate here the interpolation of two functions, $f_1(x) = \cos(x)$ and $f_2(x) = (1 - 0.9\,\sin(2\,\pi\,x))^{1/3}$ on the interval $[0,1]$, via some of the methods of this paper, as well as with equi-spaced interpolation polynomial. When performing trigonometric interpolation, with period $T = 1$, each of these functions is assumed to be periodic, with period 1 on $\mathbb{R}$. Thus, f_1 is discontinuous on the integers, whereas f_2 is in fact smooth and periodic on $\mathbb{R}$. Thus, for purposes of trigonometric interpolation, f_1 takes on its average value, $\cos(0) + \cos(1))/2$ at the integers, and we thus get large errors when interpolating this function via trigonometric polynomial with period $T = 1$ whereas we get much better accuracy when interpolating f_2 with such polynomilas. On the other hand, for purposes of polynomial interpolation, we get rather good results when interpolating f_1 on $[0,1]$ but poorer results when interpolating f_2, the larger errors in the latter case being due to the fact that f_2 has singularities close to the real axis. The Green's function basis approximation procedure behaves like a Fejér aprpoxiamtion, in that it has no overshoots, but converges very slowly.

We have demonstrated here the approximation procedure using an even number of intervals on a period, i.e., $h = T/(2N)$, and with interpolation at the "integer" points $\{j\,h\}$. We have not demonstrated the other cases i.e., with interpolation at the points $\{(j-1/2)\,h\}$, nor for the cases of integer or non-integer interpolation points for the case of $h = T/(2\,N+1)$, since our the results in these cases are similar. These other cases could, however, be important in some applications, depending on given data.

[1] End values may no longer be interpolations for the case of (2.7), since a Fourier series averages values at discontinuities.

References

[G1] **B.G. Gabdulhaev**, *Approximate Solution of Singular Integral Equations by the Method of Mechanical Quadratures*, Soviet Math., Dokl., v. 9 (1968) 329–332.

[S1] **F. Stenger**, *Numerical Methods Based on Sinc and Analytic Functions*, Springer-Verlag, New York (1993).

[S2] **F. Stenger**, *Sinc Approximation of Cauchy–Type Integrals Over Arcs*, Proceedings of a meeting in Honor of David Elliott's 65^{th} Birthday, (1998).

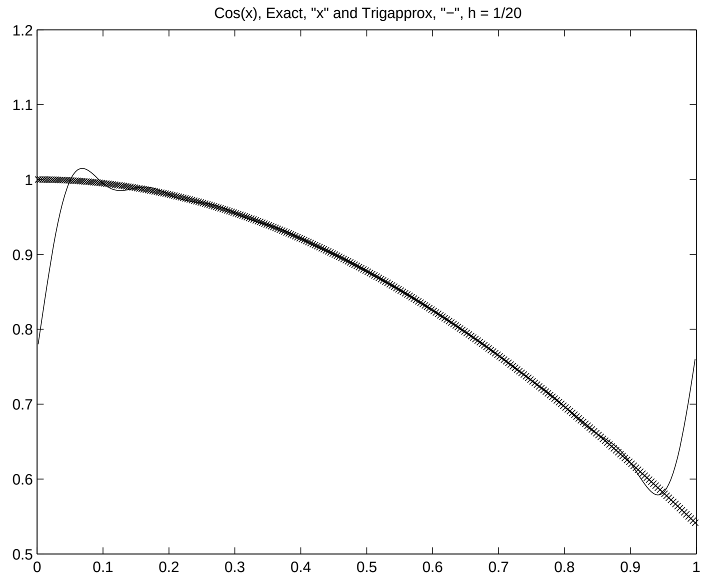

Figure 1: Cos(x) Exact, “x” and Trig approximation, “-”

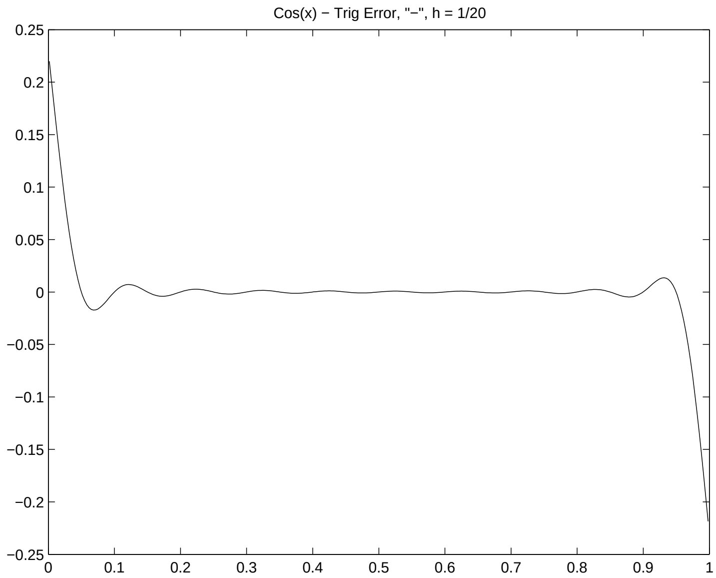

Figure 2: Error of Trig Approximation of Cos(x)

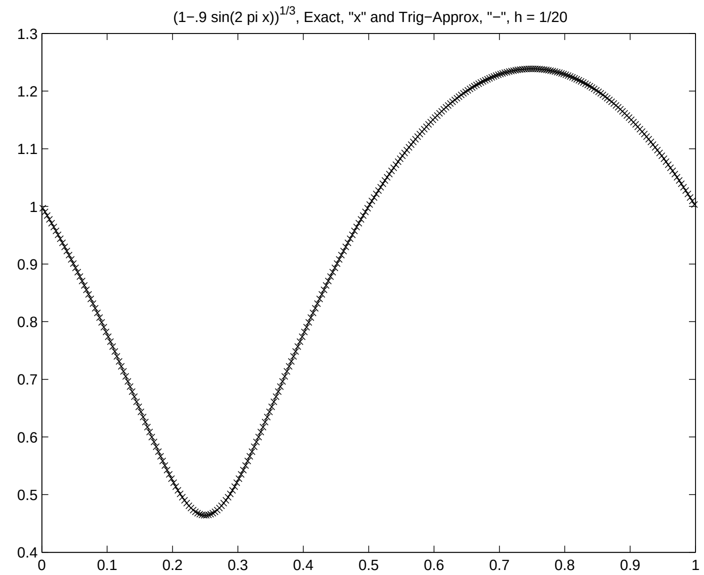

Figure 3: Exact and Trig Approximation of $(1 - .9\,\sin(2\,\pi\,x))^{1/3}$

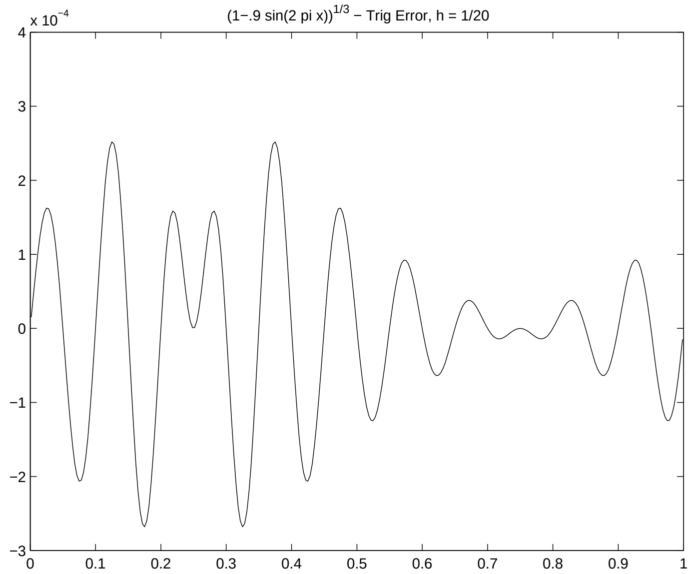

Figure 4: Error of Trig Approximation of $(1 - .9\,\sin(2\,\pi\,x))^{1/3}$

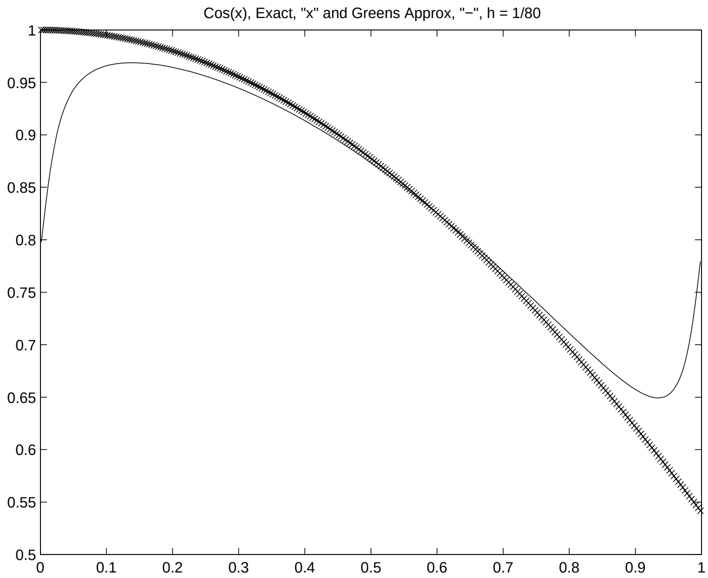

Figure 5: Exact and Green's Approximation of Cos(x)

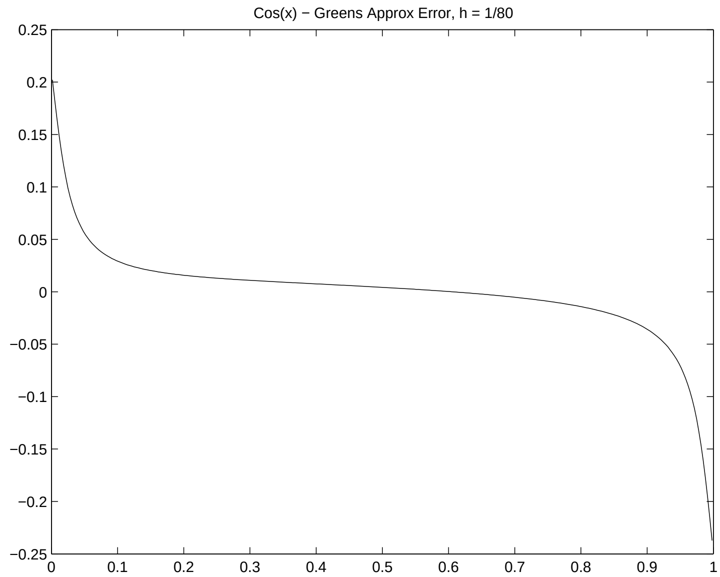

Figure 6: Error of Green's Approximation of Cos(x)

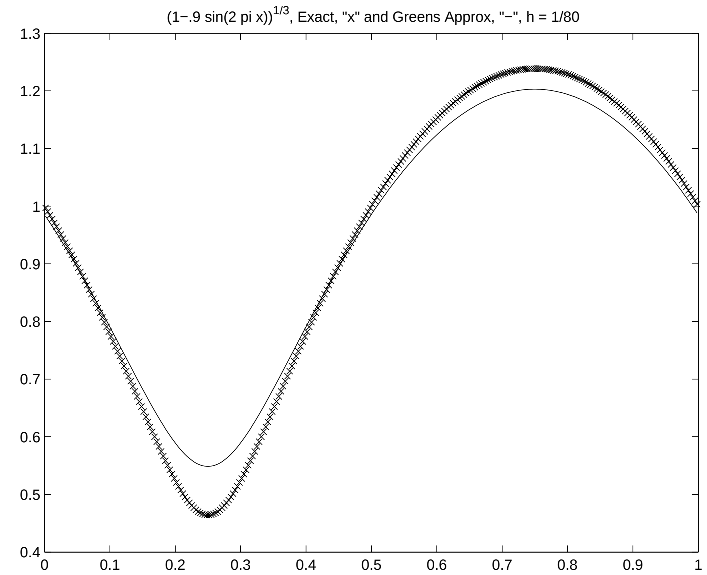

Figure 7: Exact and Green's Approximation of $(1 - .9\,\sin(2\,\pi\,x))^{1/3}$

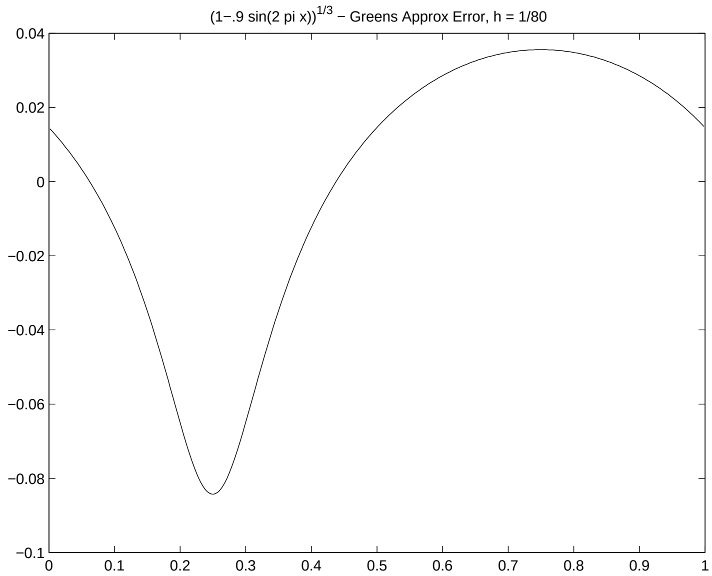

Figure 8: Error of Green's Approximation of $(1 - .9\,\sin(2\,\pi\,x))^{1/3}$

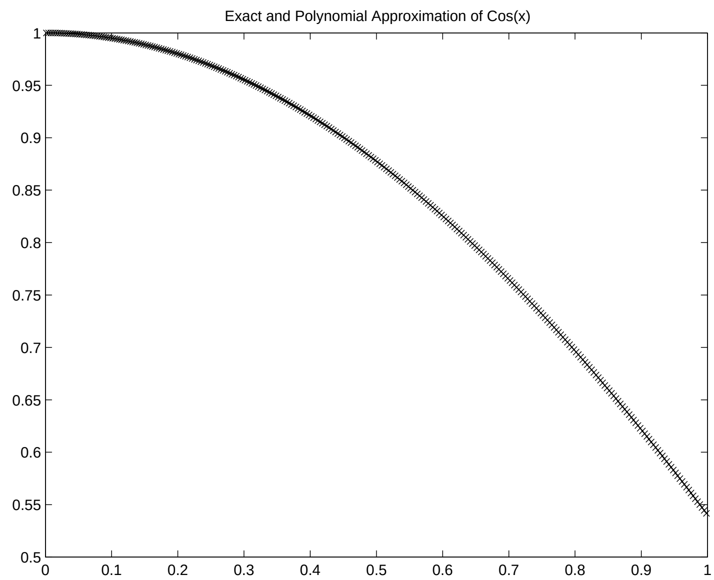

Figure 9: Exact and Polynomial Approximation of Cos(x)

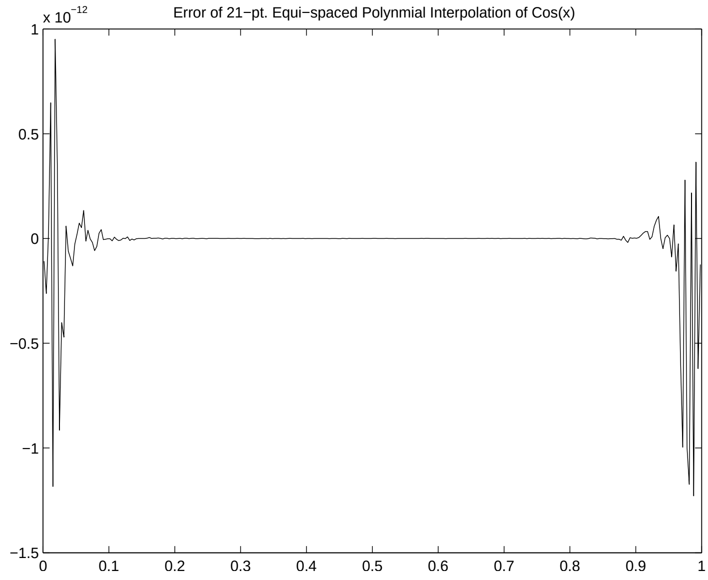

Figure 10: Error of 21-pt. Equi-spaced Polynmial Interpolation

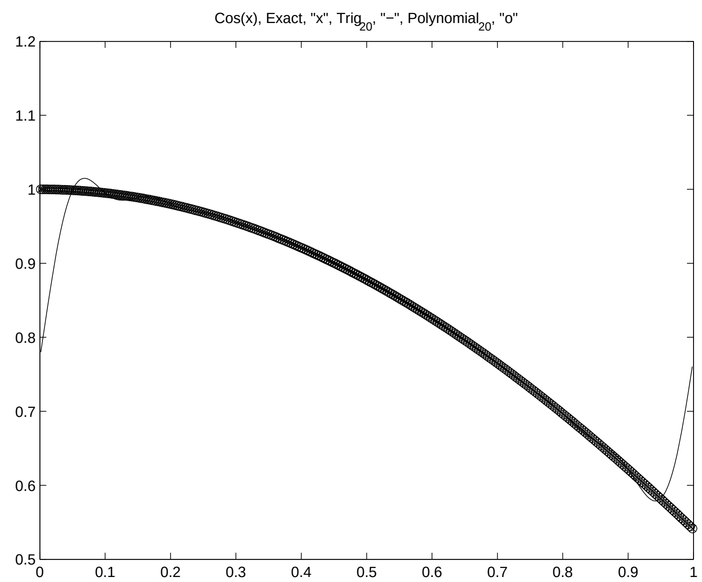

Figure 11: Exact,Trig and Polynomial Approximation of $(1 - .9\,\sin(2\,\pi\,x))^{1/3}$

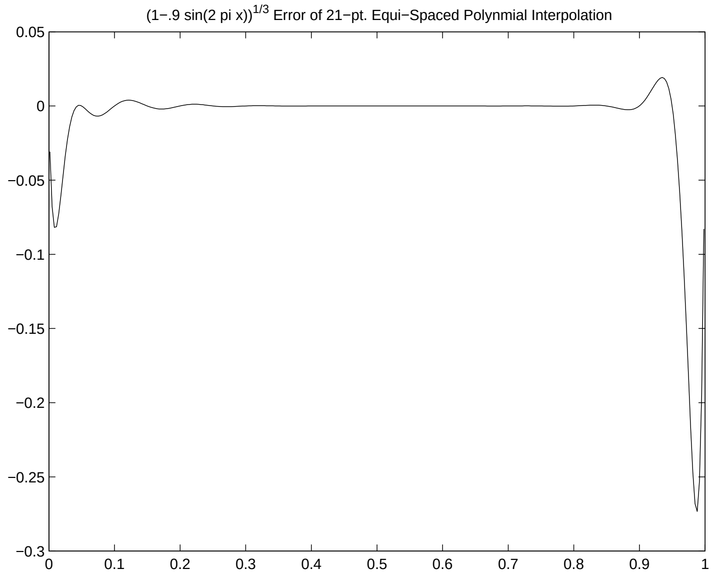

Figure 12: Error of 21 Pt Equi-Spaced Poly Approx of $(1 - .9\,\sin(2\,\pi\,x))^{1/3}$

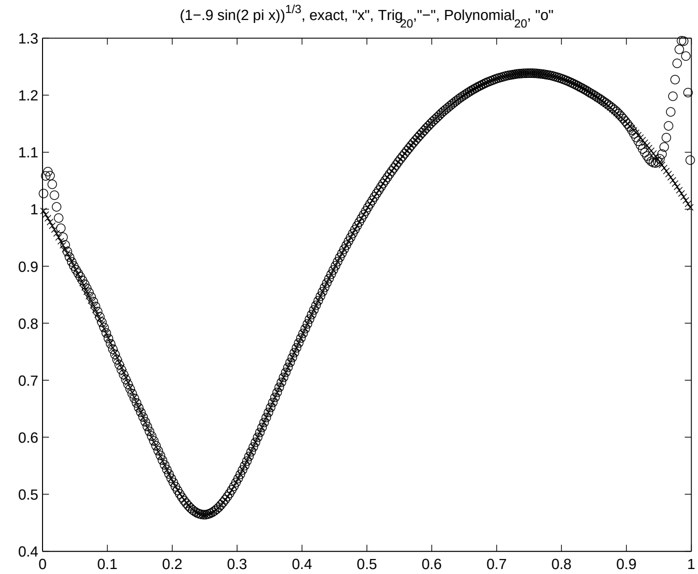

Figure 13: Exact, ‘x’, Trig(20), ‘-’, Poly(20), ‘o’, Approx of $(1 - .9\,\sin(2\,\pi\,x))^{1/3}$