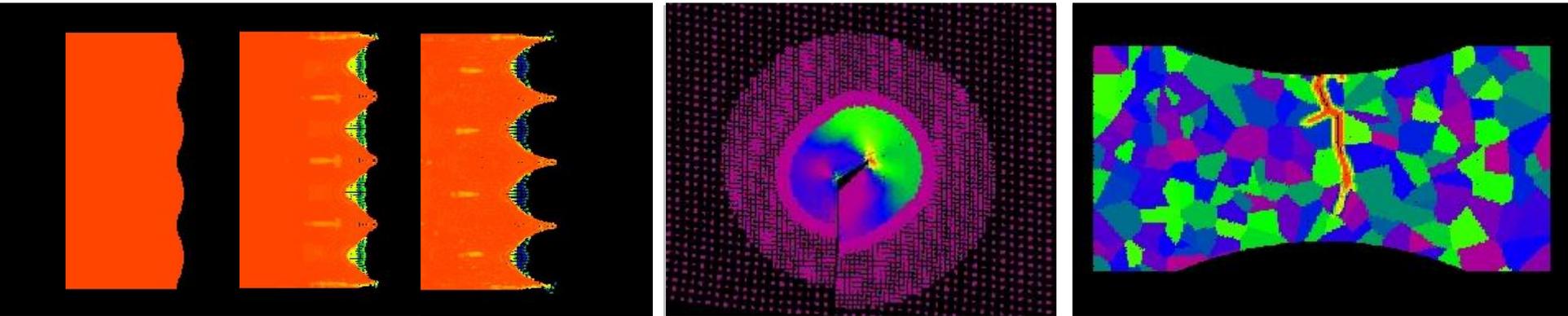


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# Origin and effect of nonlocality in a composite

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# Objectives

- Demonstrate how nonlocality arises in a simple material system.
- Derive a peridynamic model from a micromechanical model.
- Show that nonlocality makes a difference in a macroscopic experiment.

# Nonlocality and homogenization

- Weak nonlocality appears in certain approaches to homogenization.  
Examples:
  - Beran and McCoy (1970)
  - Gambin and Kroner (1989)
- Typical result is that as the periodic length scale gets small, the homogenized equations include a length scale of the same order.
- Present work:
  - Seek a mechanical view of nonlocality.
  - Don't rely on convergence or asymptotics.

# Layered composite



Two phases:

- Compliant: elastic moduli  $\mu_c, E_c$ .
- Stiff: elastic moduli  $\mu_s, E_s$ .
- $\mu_s \gg \mu_c, \quad E_s \gg E_c$
- Interfaces are rigidly connected.
- Layer thicknesses are  $2h_s, 2h_c$ .

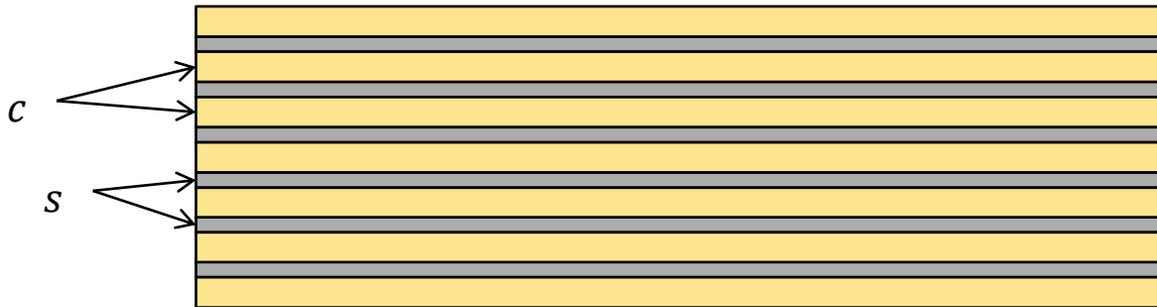
# Traditional approach

- Assume  $u_s(x) = u_c(x)$  everywhere.
- Set

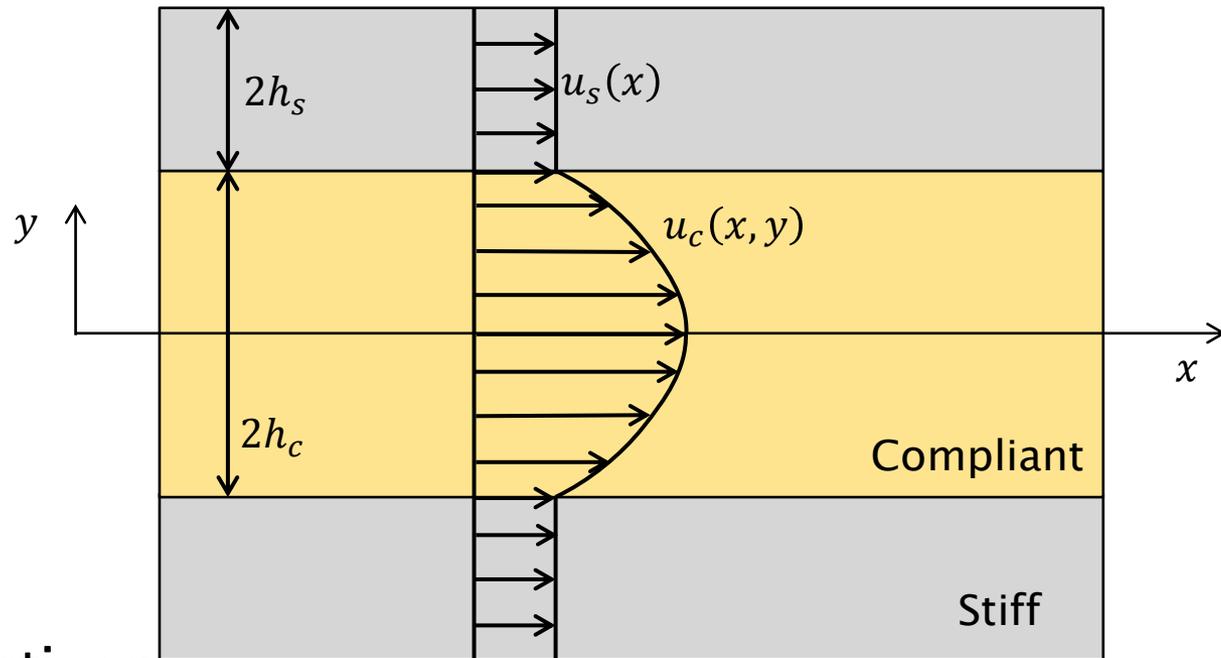
$$\bar{E} = \frac{h_s E_s + h_c E_c}{h_s + h_c}.$$

$$\bar{\sigma}'(x) + b(x) = 0, \quad \bar{\sigma} = \bar{E}u'$$

- This local “homogenized” model neglects possibility of unequal displacements between phases.
- This will turn out to be a special case of the nonlocal model to be derived.



# Micromechanical model



Assumptions:

- Set  $h_s = h_c = 1$  for simplicity.
- $u_s$  is independent of  $y$ .
- $u_c(x, y) = u_s(x) + (1 - y^2)w(x)$ .

# Equilibrium between phases

- Shear traction on a stiff layer:

$$\tau = \mu_c \frac{\partial u_c}{\partial y}(x, -1) = 2\mu_c w(x).$$

- Resulting force balance on a stiff layer:

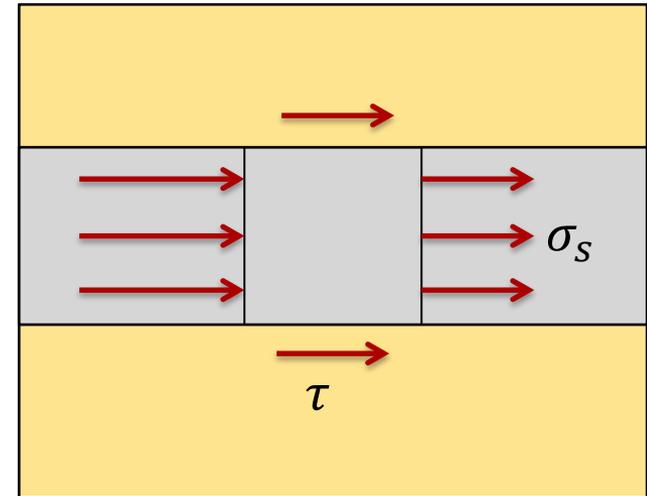
$$h_s \sigma'_s + \tau = 0$$

where  $\sigma_s$  is the normal stress in a stiff layer.

- Leads to

$$E_s u''_s(x) + 2\mu_c w(x) = 0.$$

- Now have two coupled fields:  $u_c(x)$  and  $w(x)$ .



# Smoothed displacement field

- Define

$$\bar{u}(x) = \frac{u_s(x) + u_c(x)}{2} = u_s(x) + \frac{w(x)}{3}.$$

- Recall the force balance

$$E_s u_s'' + 2\mu_c w(x) = 0.$$

- Combine the last two equations to get rid of  $u_s$ :

$$\bar{u}'' = \frac{1}{3}w''(x) - \frac{2\mu_c}{E_s}w(x).$$

# Solve ODE for $w$

- For prescribed  $\bar{u}$ , now have a 2nd order linear ODE for  $w$ :

$$\bar{u}'' = \frac{1}{3}w''(x) - \frac{2\mu_c}{E_s}w(x).$$

- Solution is

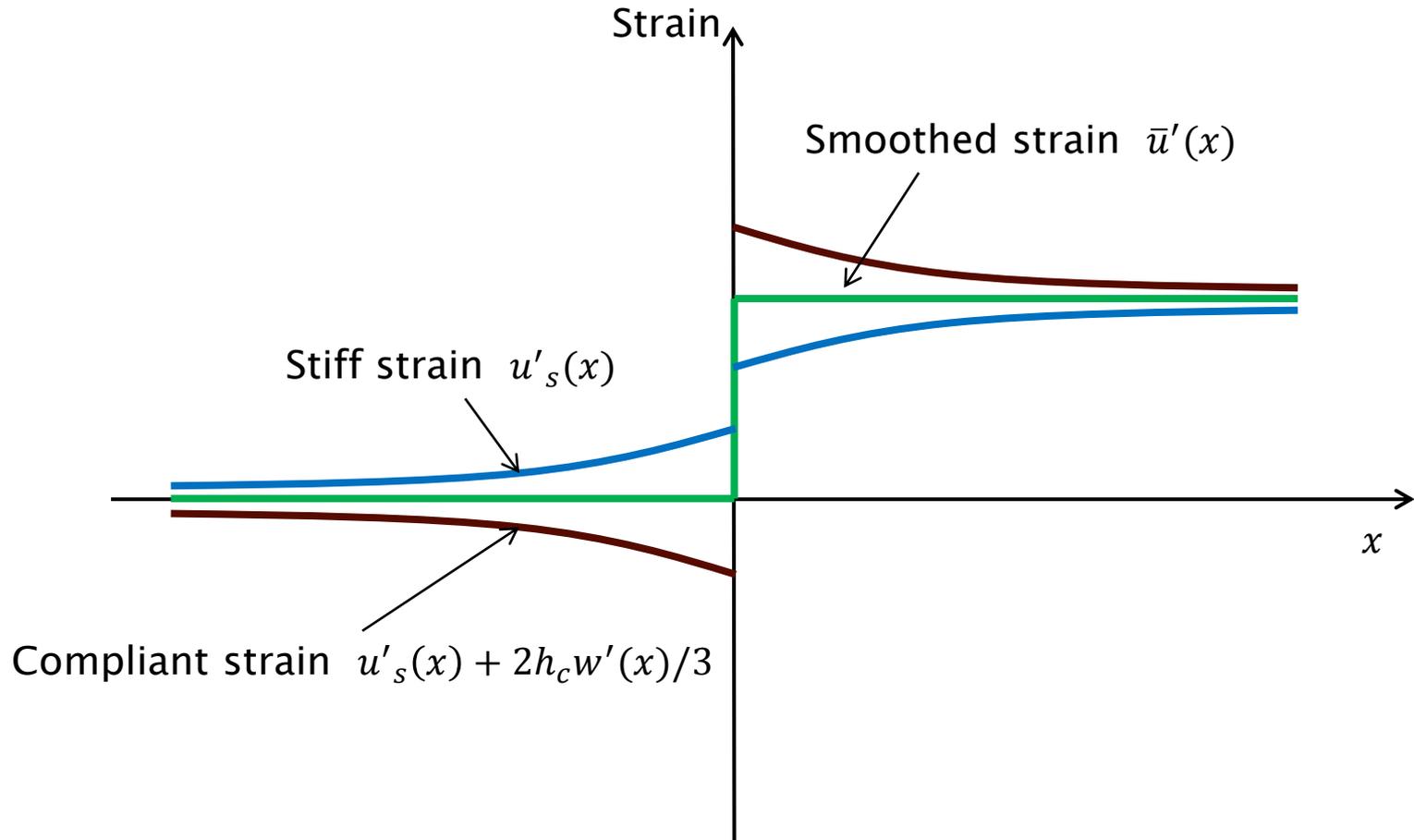
$$w(x) = - \int_{-\infty}^{\infty} \bar{u}''(p)G(x-p) dp$$

Where

$$G(x) = ke^{-\lambda|x|}, \quad \lambda = \sqrt{\frac{6\mu_c}{E_s}}, \quad k = \frac{1}{2\lambda}.$$

# Jump in smoothed strain

Set  $\bar{u}''(x) = \Delta(x)$  where  $\Delta$  is the Dirac delta function.



# Find smoothed stress

- Define

$$\bar{\sigma}(x) = \frac{1}{2} \left( \int_0^1 \sigma_s dy + \int_0^1 \sigma_c(x, y) dy \right)$$

where

$$\sigma_s = E_s \frac{\partial u_s}{\partial x}, \quad \sigma_c = E_c \frac{\partial u_c}{\partial x}.$$

- After using previous results, find

$$\bar{\sigma}(x) = \bar{E} \bar{u}'(x) + \gamma \int_{-\infty}^{\infty} \bar{u}''(p) G'(x-p) dp,$$

Local term   Nonlocal term

$$\bar{E} = \frac{E_s + E_c}{2}, \quad \gamma = \frac{E_s - E_c}{6}.$$

# Equation of motion in terms of smoothed displacement

- Get rid of  $\bar{\sigma}$ , using

$$\rho \ddot{u}(x, t) = \bar{\sigma}'(x, t) + b(x, t).$$

- After some manipulations, find

$$\rho \ddot{u}(x, t) = E_c \bar{u}''(x, t) + \gamma k \lambda^4 \int_{-\infty}^{\infty} (\bar{u}(p, t) - \bar{u}(x, t)) e^{-\lambda|x-p|} dp + b(x, t),$$

- This contains both local and nonlocal terms, similar to Kroner's nonlocal model (also DiPaola's).

# Can we avoid the local term?

- Observe that for any function  $v(x)$ ,

$$v''(x) = \int_{-\infty}^{\infty} v(p) \Delta''(x-p) dp.$$

- Approximate

$$\Delta(x) \approx \frac{\tau e^{-\tau|x|}}{2}$$

where  $\tau$  is a large parameter.

- Replace the local term in the equation of motion to arrive at

$$\rho \ddot{u}(x, t) = \int_{-\infty}^{\infty} \left( \frac{E_c \tau}{2} e^{-\tau|x-p|} + \gamma k \lambda^4 e^{-\lambda|x-p|} \right) (\bar{u}(p, t) - \bar{u}(x, t)) dp + b(x, t),$$

- This is the equation of motion in linearized peridynamics.

# Nonlocal length scale

- Recall

$$\rho \ddot{u}(x, t) = \int_{-\infty}^{\infty} \left( \frac{E_c \tau}{2} e^{-\tau|x-p|} + \gamma k \lambda^4 e^{-\lambda|x-p|} \right) (\bar{u}(p, t) - \bar{u}(x, t)) dp + b(x, t),$$

 Physically nonlocal term

$$\frac{1}{\lambda} = \sqrt{\frac{E_s}{6\mu_c}} = \text{length scale if } h_s = h_c = 1.$$

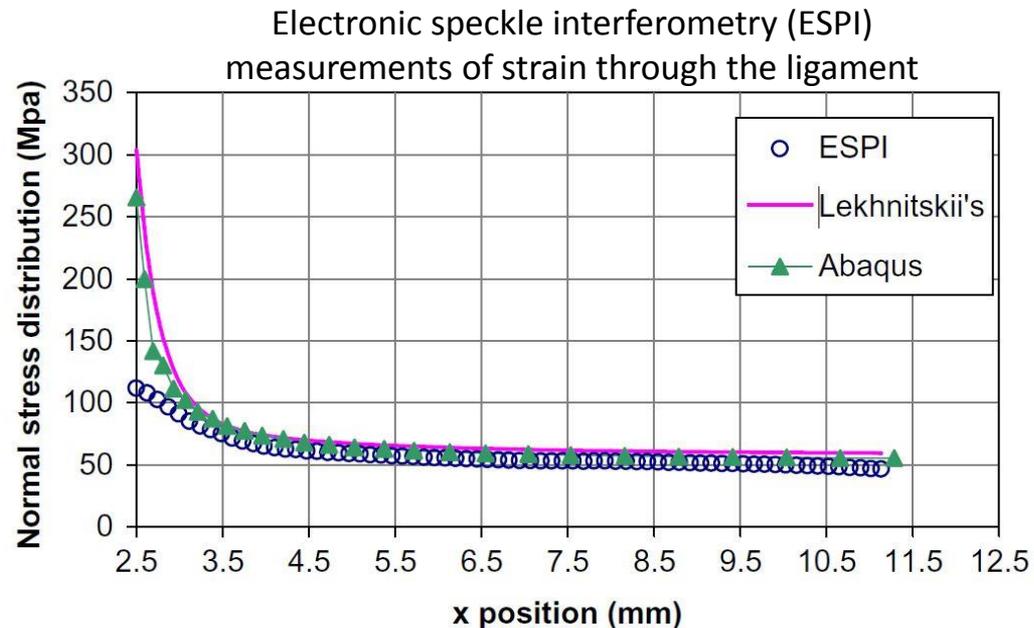
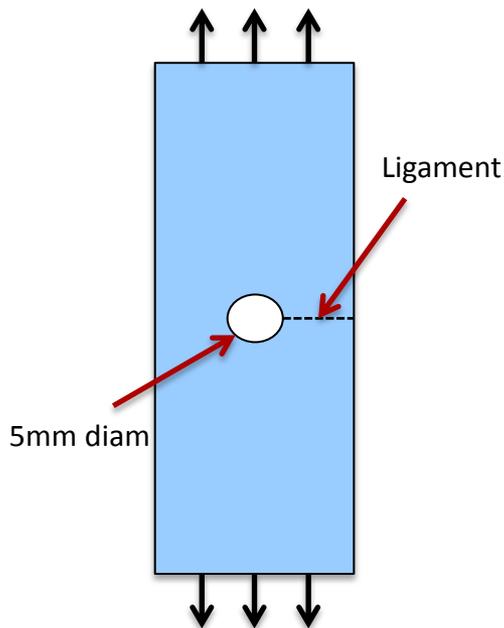
- More generally

$$\frac{1}{\lambda} = \sqrt{\frac{E_s h_s h_c^2}{3\mu_c (h_s + h_c)}} = \text{length scale.}$$

- The softer the compliant material is, the greater the length scale.
- The nonlocal length scale is not determined only by the geometrical scales.

# Are composites nonlocal?

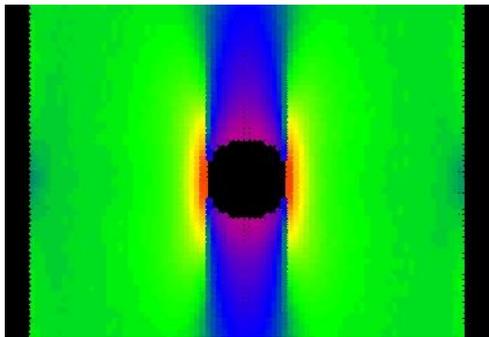
- Open hole tension in a fabric laminate:
  - FEM agrees with anisotropic local theory in predicting the stress concentration near the hole.
  - Both overpredict the stress concentration.



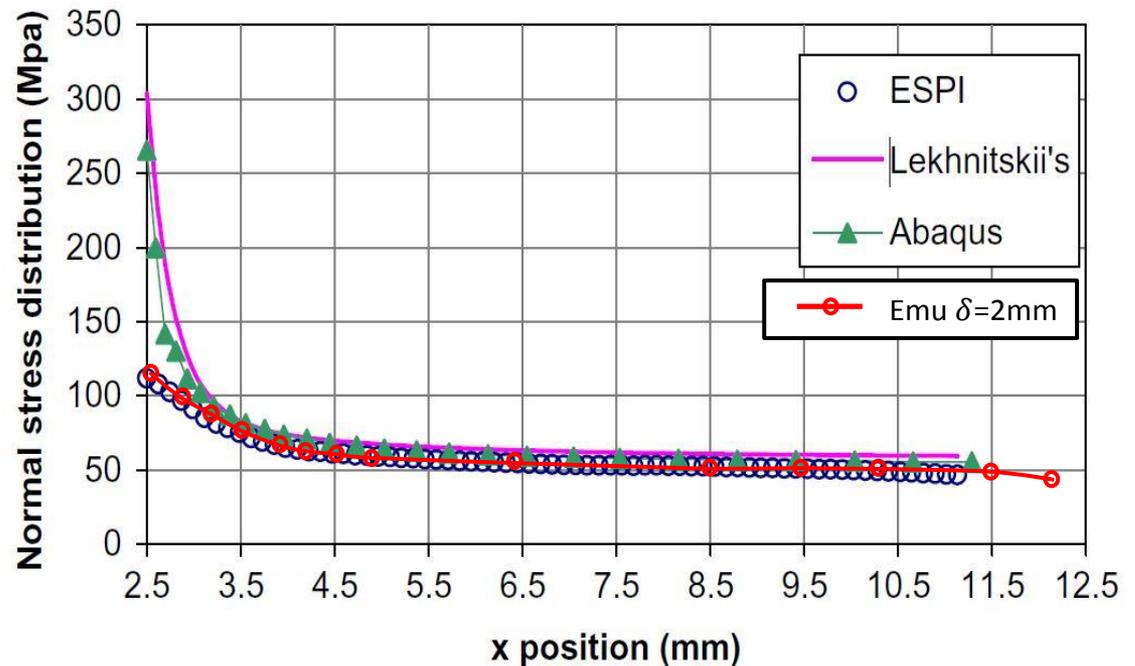
Source: Toubal, Karama, and Lorrain, Composite Structures 68 (2005) 31-36

# Are composites nonlocal?

- Peridynamic model is more accurate than the local model for predicting stress concentration in a laminate.
- $h_s = h_c = 0.4\text{mm}$ ,  $E_s = 150\text{GPa}$ ,  $\mu_c = 4\text{GPa}$ .
- $\Rightarrow 1/\lambda = 1.41\text{mm}$ .



EMU: contours of longitudinal stress  
Horizon = 2mm



Data of Toubal, Karama, and Lorrain, Composite Structures 68 (2005) 31-36

# Discussion

- The modeling decision to use a smoothed displacement field introduced nonlocality.
  - Constraining the mean displacement  $\bar{u}$  allows the phases to wiggle relative to each other.
- The nonlocal term vanishes if the strain is uniform.
  - It becomes important if strong gradients are present.
- The length scale for nonlocality involves the material properties as well as the geometrical length scales.