A New Metric On the Complex Numbers
Sand No. 98-1754 *

David Day

August 13, 1998

Abstract

This work proves that

$$\varrho_\infty(a,b) = \frac{|a-b|}{\max(|a|,|b|)}$$

is a metric on the complex numbers with the understanding that $0/0 = 0$.

1 Background and Proof

The function $\varrho_\infty(.,.)$ arises in the analyses of quantities computed in floating point arithmetic with machine precision ϵ. A complex floating point number, b, approximates an unreal number, a, in the sense that $\varrho_\infty(a,b) \le \epsilon$. We prove that

$$\varrho_\infty(a,b) + \varrho_\infty(b,c) - \varrho_\infty(a,c) \ge 0$$

by determining that the critical points of a normalized problem are real.

Unless $|b| > \max(|a|,|c|)$ the triangle inequality for $\varrho_\infty(.,.)$ follows from the ordinary triangle inequality. It suffices to consider the case $|b| \ge |a| \ge |c|$. We rotate a, b, c so that $b = |b|$, and then scale such that $b = 1$. Next we introduce notation for polar coordinates,

$$a = \alpha e^{i\phi} \quad \text{and} \quad c = \gamma e^{i\theta}.$$

Note that the triangle inequality holds in the obvious cases $1 = a$ or $a = c$ or $a = 0$ or $c = 0$, and henceforth we assume these equations do not hold. We also introduce

$$f(\sigma,\psi) = |1 - \sigma e^{i\psi}|$$

because $\varrho_\infty(a,1) = f(\alpha,\phi), \varrho_\infty(1,c) = f(\gamma,\theta)$, and

$$\varrho_\infty(a,c) = f(\frac{\gamma}{\alpha}, \theta - \phi).$$

We proceed from the observation that because f is smooth away from $\sigma = 0$, it is possible to use elementary techniques to minimize

$$T(\alpha,\phi,\gamma,\theta) = f(\alpha,\phi) + f(\gamma,\theta) - f(\frac{\gamma}{\alpha}, \theta - \phi)$$

*Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

on $\mathcal{D} = \{\alpha, \phi, \gamma, \theta : 0 < \gamma \leq \alpha <= 1, 0 \leq \phi, \theta < 2\pi\}$.

To compute ∇T, differentiate

$$f^2 = 1 - 2\sigma\cos(\psi) + \sigma^2$$

first w.r.t ψ and second w.r.t σ, and divide by $2f$ to find

$$D_\psi f = \sigma\sin(\psi)/f, \quad \text{and} \quad D_\sigma f = (\sigma - \cos(\psi))/f.$$

This identity and the chain rule imply that

$$D_\phi T = \frac{\alpha\sin(\phi)}{\varrho_\infty(a,1)} + \frac{\gamma}{\alpha}\frac{\sin(\theta-\phi)}{\varrho_\infty(a,c)},$$

and that

$$D_\theta T = \frac{\gamma\sin(\theta)}{\varrho_\infty(1,c)} - \frac{\gamma}{\alpha}\frac{\sin(\theta-\phi)}{\varrho_\infty(a,c)}.$$

Similarly,

$$D_\alpha T = \frac{\alpha - \cos(\phi)}{\varrho_\infty(a,1)} + \frac{\gamma}{\alpha^2}\frac{\gamma/\alpha - \cos(\theta-\phi)}{\varrho_\infty(a,c)},$$

and

$$D_\gamma T = \frac{\gamma - \cos(\theta)}{\varrho_\infty(1,c)} - \frac{1}{\alpha}\frac{\gamma/\alpha - \cos(\theta-\phi)}{\varrho_\infty(a,c)}.$$

$\nabla T = \mathbf{0}$ at $(\alpha, \phi, \gamma, \theta) \in \mathcal{D}$ for which

$$-\frac{\alpha^2}{\gamma}\frac{\sin(\phi)}{\varrho_\infty(a,1)} = \frac{\sin(\theta-\phi)}{\varrho_\infty(a,c)} = \frac{\alpha\sin(\theta)}{\varrho_\infty(1,c)},$$

and

$$-\frac{\alpha^2}{\gamma}\frac{\alpha-\cos(\phi)}{\varrho_\infty(a,1)} = \frac{\gamma/\alpha - \cos(\theta-\phi)}{\varrho_\infty(a,c)} = \alpha\frac{\gamma - \cos(\theta)}{\varrho_\infty(1,c)}.$$

Next we solve these equations for the critical points.

First we extract and eliminate ratios of values of $\varrho_\infty(,.,)$:

$$-\frac{\alpha^2}{\gamma}\frac{\sin(\phi)}{\sin(\theta-\phi)} = \frac{\varrho_\infty(a,1)}{\varrho_\infty(a,c)} = -\frac{\alpha^2}{\gamma}\frac{\alpha-\cos(\phi)}{\gamma/\alpha - \cos(\theta-\phi)},$$

$$\frac{1}{\alpha}\frac{\sin(\theta-\phi)}{\sin(\theta)} = \frac{\varrho_\infty(a,c)}{\varrho_\infty(1,c)} = \frac{1}{\alpha}\frac{\gamma/\alpha - \cos(\theta-\phi)}{\gamma - \cos(\theta)},$$

$$-\frac{\gamma}{\alpha}\frac{\sin(\theta)}{\sin(\phi)} = \frac{\varrho_\infty(1,c)}{\varrho_\infty(a,1)} = -\frac{\gamma}{\alpha}\frac{\gamma-\cos(\theta)}{\alpha-\cos(\phi)}.$$

The first equations becomes

$$(\frac{\gamma}{\alpha} - \cos(\theta-\phi))\sin(\phi) = (\alpha - \cos(\phi))\sin(\theta-\phi).$$

Using $\sin(\phi - (\theta-\phi)) = \sin(\phi)\cos(\theta-\phi) - \cos(\phi)\sin(\theta-\phi)$, this simplifies to

$$\frac{\gamma}{\alpha}\sin(\phi) = \alpha\sin(\theta-\phi) - \sin(2\phi-\theta).$$

Similar analyses of the two equations that remain produce

$$\frac{\gamma}{\alpha}\sin(\theta) = \gamma\sin(\theta-\phi) + \sin(\phi),$$

and

$$\gamma\sin(\phi)-\alpha\sin(\theta)=\sin(\theta-\phi).$$

Second we eliminate γ from these equations:

$$\begin{array}{ccccc}\gamma & = & \dfrac{\alpha}{\sin(\phi)}(\alpha\sin(\theta-\phi)-\sin(2\phi-\theta)) & = \\ \dfrac{\alpha\sin(\phi)}{\sin(\theta)-\alpha\sin(\theta-\phi)} & = & \dfrac{\alpha\sin(\theta)+\sin(\theta-\phi)}{\sin(\phi)}.\end{array}$$

Third solve the first and third of these equations for α:

$$\alpha^2-\frac{\sin(2\phi-\theta)+\sin(\theta)}{\sin(\theta-\phi)}\alpha-1=0.$$

Also combine the first and second equations and one has

$$\alpha^2-\frac{\sin(2\phi-\theta)+\sin(\theta)}{\sin(\theta-\phi)}\alpha+\frac{\sin^2(\phi)+\sin(\theta)\sin(2\phi-\theta)}{\sin^2(\theta-\phi)}=0.$$

Since for fixed θ and ϕ these two polynomials differ by a constant, they share a root if and only if they are equal:

$$\frac{\sin^2(\phi)+\sin(\theta)\sin(2\phi-\theta)}{\sin^2(\theta-\phi)}=-1.$$

This equation first simplifies to

$$\sin^2(\phi)+\sin(\theta)\sin(2\phi-\theta)+\sin^2(\theta-\phi)=0.$$

We use the expansion

$$\sin(2\phi-\theta)\sin(\theta)=(\ 2\cos(\phi)\sin(\phi)\ )\cos(\theta)\sin(\theta)-\cos(2\phi)\sin^2(\theta)$$

to decipher this expression:

$$\sin^2(\phi)+\sin(\theta)\sin(2\phi-\theta)+\sin^2(\theta-\phi)=$$

$$-\cos(2\phi)\sin^2(\theta)+(\sin(\theta)\cos(\phi))^2+(-\cos(\theta)\sin(\phi))^2=\sin^2(\phi).$$

At a critical point $\sin(\phi)=0$. But then $\sin(\theta)=0$ must also hold for $D_\phi T=0$. That is, a and c are real.

The critical points in the real case with $1=b\geq|a|\geq|c|$ lie along the kernal of f, $a=c$, in which case

$$\varrho_\infty(a,b)+\varrho_\infty(b,c)-\varrho_\infty(a,c)=\varrho_\infty(a,b)+\varrho_\infty(b,c)\geq 0.$$

This completes the proof that $\varrho_\infty(.,.)$ is a metric on $\mathcal{C}$.