# Localized Coarsening of Conforming All-Hexahedral Meshes

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**Abstract:** Finite element mesh adaptation methods can be used to improve the efficiency and accuracy of solutions to computational modeling problems. In many applications involving hexahedral meshes, localized modifications which preserve a conforming all-hexahedral mesh are desired. Effective hexahedral refinement methods that satisfy these criteria have recently become available; however, due to hexahedral mesh topology constraints, little progress has been made in the area of hexahedral coarsening. This paper presents a new method to locally coarsen conforming all-hexahedral meshes. The method works on both structured and unstructured meshes and is not based on undoing previous refinement. Building upon recent developments in quadrilateral coarsening, the method utilizes hexahedral sheet and column operations, including pillowing, column collapsing, and sheet extraction. A general algorithm for automated coarsening is presented and examples of models that have been coarsened with this new algorithm are shown. While results are promising, further work is needed to improve the automated process.

**Keywords: Hexahedral, Mesh, Coarsening, Simplification, Adaptivity, Refinement**

## 1 Introduction

The efficiency and accuracy of a finite element computational modeling solution are greatly influenced by the distribution of elements in the finite element mesh. In a given model, there are usually regions that require greater mesh density than others. A higher concentration of elements in these regions may be necessary to reduce error in the finite element approximation, increase resolution where there are high gradients, or more accurately represent the model geometry. Regions where high accuracy is not critical or where gradients are low can generally be modeled with lower mesh density. Since the computational time required in a finite element analysis is directly related to the number of elements in the model being analyzed, it is advantageous to produce a mesh that has as few elements as possible. Therefore, in an ideal analysis, each region in the model should have enough elements to produce a good solution, but no more.

For most real-world finite element models, current mesh generation algorithms are unable to create an initial mesh that optimizes both accuracy and efficiency in the finite element solution. Although some control over mesh density is possible, an initial mesh will almost always contain regions that have too few elements, regions that have too many elements, or both. In addition, some finite element applications require mesh density to evolve throughout an analysis as areas of high and low activity change with time [1, 2, 3, 4]. For these reasons, much research has been devoted to the development of mesh adaptation tools that make it possible to adjust element density in specific regions either before or during analysis.

Mesh adaptation consists of both refinement and coarsening. Refinement is the process of increasing mesh density by adding elements to a mesh, while coarsening is the process of decreasing mesh density by removing elements from a mesh. By refining areas that have too few elements and coarsening areas that have too many elements, a more accurate and efficient analysis can be performed.

To satisfy the requirements of some finite element solvers, a mesh must be topologically conforming and contain only one element type. In general, a conforming all-tetrahedral mesh can be locally modified with much greater ease than a conforming all-hexahedral mesh. However, in many modeling applications, hexahedral elements are preferred over tetrahedral elements because they provide greater efficiency and accuracy in the computational process [1, 5]. For this reason, work has been done to improve hexahedral mesh adaptation methods. As a result, robust hexahedral refinement algorithms are becoming available [6, 7, 8, 9]. However, few developments have been seen in the area of hexahedral coarsening. The lack of effective coarsening methods creates a major gap in the field of hexahedral mesh adaptation.

To effectively achieve the objectives of mesh adaptation, a truly general hexahedral coarsening algorithm should:

1. Preserve a conforming all-hexahedral mesh
2. Restrict mesh topology and density changes to defined regions
3. Work on both a structured and unstructured mesh
4. Not be limited to only undoing previous refinement

Although hexahedral coarsening has been utilized in some modeling applications, no single algorithm has been developed that satisfies all the criteria listed above. This is, in large part, due to the topology constraints that exist in a conforming all-hexahedral mesh. These constraints make it difficult to modify mesh density without causing topology changes to propagate beyond the boundaries of a defined region [10, 11].

Since current hexahedral coarsening methods are unable to satisfy all the requirements listed above, they have limited applications. For example, to prevent global topology changes, some algorithms introduce non-conforming or non-hexahedral elements into the mesh [1, 2, 12, 13, 14]. While this is a valid solution for some types of analysis, not all finite element solvers can accommodate hanging nodes or hybrid meshes. Other algorithms maintain a conforming all-hexahedral mesh, but they generally require either global topology changes beyond the defined coarsening region [11, 15, 16], structured mesh topology where predetermined transition templates can be used [8, 15], or prior refinement that can be undone [2, 12, 13]. These weaknesses severely limit the effectiveness of these algorithms on most real-world models.

This paper presents a new method to locally coarsen conforming all-hexahedral meshes. The method works on both structured and unstructured meshes and is not based on undoing previous refinement. The remainder of this paper is organized as follows. Section 2 provides an overview of some basic hexahedral mesh operations. Section 3 shows how these operations have been combined to produce localized hexahedral coarsening and how the coarsening process has been automated. In Section 4, some examples of models which have been coarsened are shown. Finally, in Section 5, some areas of future work are discussed.

## 2 Hexahedral Mesh Operations

In recent years, a greater understanding of hexahedral mesh topology has led to the development of many new hexahedral mesh operations [17, 18, 19]. In this section, three operations which are useful for hexahedral coarsening are presented. These operations are based on hexahedral sheets and columns, which are topology-based groups of hexahedra that always exist in a conforming hexahedral mesh.

### 2.1 Hexahedral Sheets and Columns

A hexahedral element contains three sets of four topologically parallel edges, as shown in Figure 1. Topologically parallel edges provide the basis for hexahedral sheets. The formation of a sheet begins with a single edge. Once an edge has been chosen, all elements which share that edge are identified. For each of these elements, the three edges which are topologically parallel to the original edge are also identified. These new edges are then used to find another layer of elements and topologically parallel edges. This process is repeated until no new adjacent elements can be found. The set of elements which are traversed during this process makes up a hexahedral sheet. Figure 2 shows a hexahedral mesh with one of the sheets in the mesh defined.

A hexahedral element also contains three pairs of topologically opposite quadrilateral faces, as shown in Figure 3. Topologically opposite faces provide

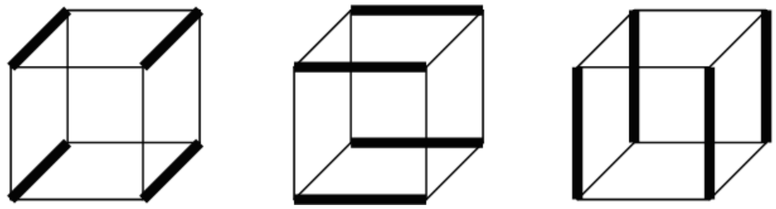

**Fig. 1.** A hexahedral element's three sets of topologically parallel edges.

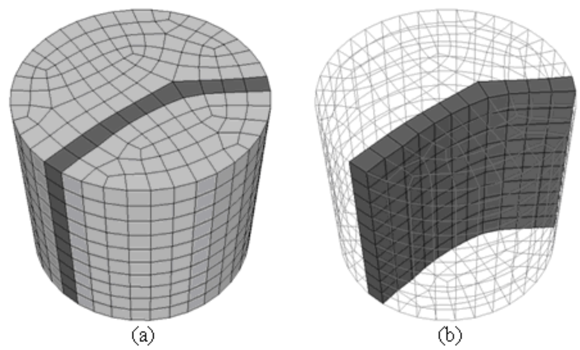


**Fig. 2.** A hexahedral sheet: (a) A hexahedral mesh with one sheet defined. (b) A view of the entire sheet.

the basis for hexahedral columns. The formation of a column begins with a single face. Once a face has been chosen, the elements which share that face are identified. For each of these elements, the face which is topologically opposite of the original face is also identified. These new faces are then used to find another layer of elements and topologically opposite faces. This process is repeated until no new adjacent elements can be found. The set of elements which are traversed during this process makes up a hexahedral column. An important relationship between sheets and columns is that a column defines the intersection of two sheets. This relationship is illustrated in Figure 4.

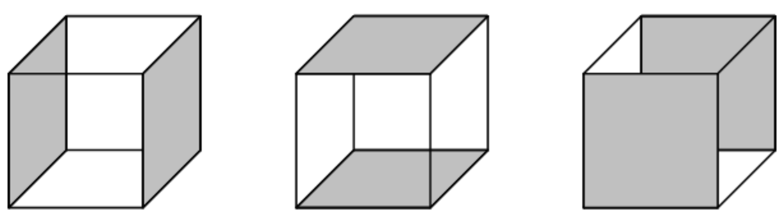

**Fig. 3.** A hexahedral element's three pairs of topologically opposite faces.

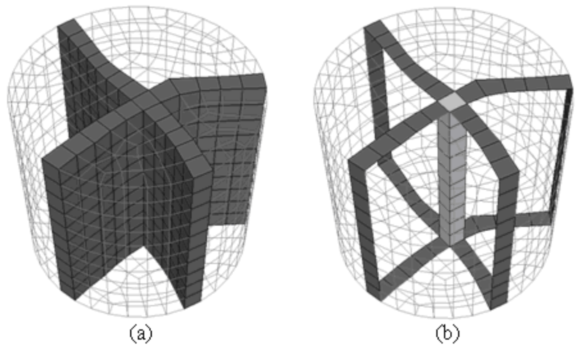


**Fig. 4.** A hexahedral column: (a) Two intersecting sheets. (b) The column that defines the intersection of the two sheets in (a).

### 2.2 Sheet and Column Operations

Hexahedral sheet and column operations can be used to modify a hexahedral mesh without introducing non-conforming elements. One such operation is known as sheet extraction [16]. Sheet extraction removes a sheet from a mesh by simply collapsing the edges that define the sheet and merging the two nodes on each edge, as shown in Figure 5. Merging nodes in this manner decreases element density in the vicinity of the extracted sheet and guarantees that the resulting mesh will be conforming.

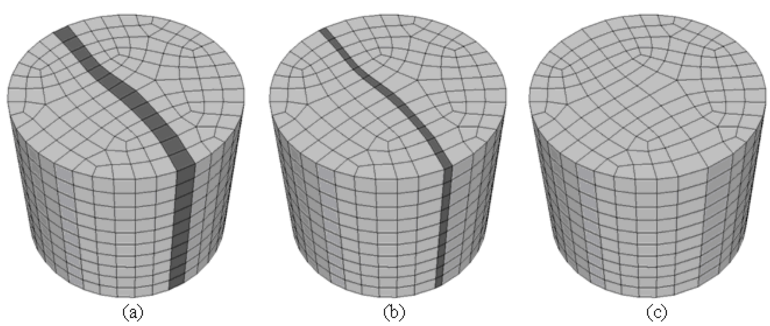


**Fig. 5.** Sheet extraction: (a) A sheet is selected for extraction. (b) The edges that define the sheet are collapsed. (c) The two nodes on each edge are merged, which eliminates the sheet and preserves a conforming hexahedral mesh.

Another hexahedral mesh operation that involves sheets is pillowing [19, 20]. Unlike sheet extraction, which removes an existing sheet from a mesh, pillowing inserts a new sheet into a mesh. As demonstrated in Figure 6, pillowing is performed on a contiguous group of hexahedral elements which make up a 'shrink' set. These elements are reduced in size and pulled away from the rest of the mesh, leaving a gap. A new sheet is then inserted into the gap by reconnecting each of the separated node pairs with a new edge. The new sheet increases element density in the vicinity of the shrink set and ensures the preservation of a conforming hexahedral mesh.

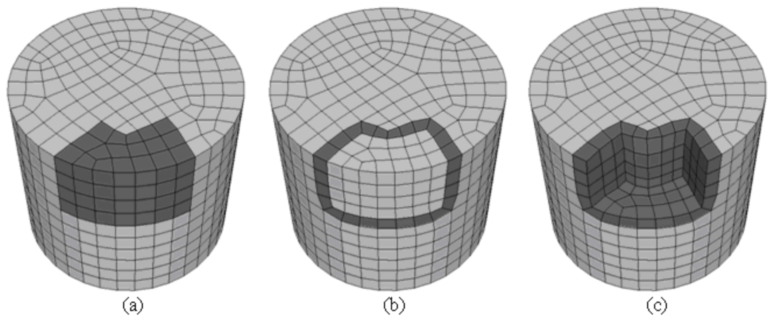


**Fig. 6.** Pillowing: (a) A shrink set is defined. (b) The elements in the shrink set are reduced in size and separated from the rest of the mesh. A sheet is inserted to fill in the gap and preserve a conforming hexahedral mesh. (c) The newly inserted sheet.

A third hexahedral mesh operation is known as column, or face, collapsing [18, 21]. A column is collapsed by merging diagonally opposite nodes in each quadrilateral face that defines the column, as shown in Figure 7. Since a quadrilateral face has two pairs of diagonally opposite nodes, a column can be collapsed in one of two different directions.

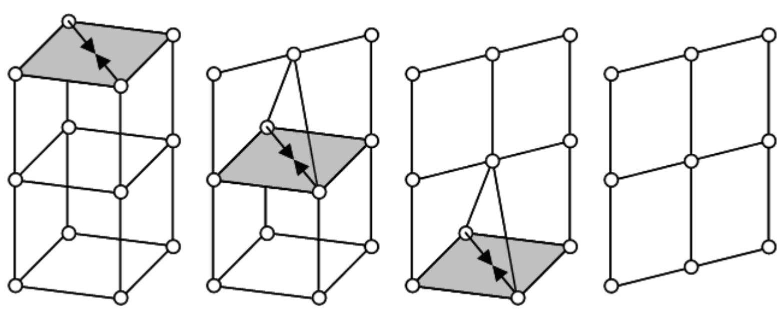

**Fig. 7.** Column collapse operation.

As previously mentioned, a column defines the intersection of two sheets. When a column is collapsed, two intersecting sheets are altered such that they no longer intersect, as illustrated in Figure 8. The paths of the new sheets are determined by the direction of the collapse. Just like sheet extraction and pillowing, the column collapse operation always preserves a conforming hexahedral mesh. In addition, similar to sheet extraction, the column collapse operation decreases element density in the vicinity of the collapsed column.

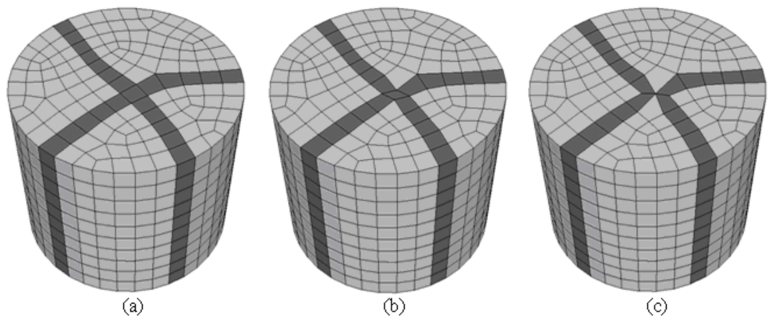


**Fig. 8.** Redirection of intersecting sheets through column collapsing: (a) Two intersecting sheets. (b) The column defining the intersection is collapsed. (c) The two sheets no longer intersect.

## 3 Hexahedral Mesh Coarsening

Utilizing the sheet and column operations described in Section 2, the hexahedral coarsening method presented in this section builds upon recent developments in quadrilateral coarsening [21]. While it is true that some quadrilateral coarsening operations can be directly extended to hexahedral coarsening, by themselves, these operations are not always able to prevent changes in element density from propogating beyond the boundaries of a defined hexahedral coarsening region.

### 3.1 Previously Developed Coarsening Techniques

As illustrated in Section 2.2, sheet extraction decreases mesh density by removing elements from a mesh. Therefore, sheet extraction is a very useful tool for hexahedral coarsening. However, sheet extraction by itself is generally not sufficient when localized coarsening is desired. This is due to the fact that sheets are rarely contained entirely within a region that has been selected for coarsening. As shown in Figure 9, extracting a sheet that extends beyond the boundaries of a defined region decreases mesh density in areas where coarsening is not desired. Therefore, before sheet extraction can occur, it is often necessary to modify the mesh in such a way that produces sheets which are contained entirely within the boundaries of a defined coarsening region.

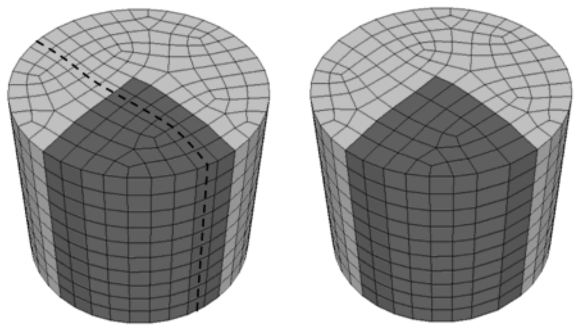


**Fig. 9.** Global coarsening: (a) A sheet passes through a region selected for coarsening. (b) When the sheet is extracted, mesh density is decreased both inside and outside the defined coarsening region.

As described in Section 2.2, the paths of intersecting sheets can be altered using the column collapse operation. Figure 10, shows how this operation can be used to create a sheet that is contained entirely within a defined region. Such a sheet can then be extracted to coarsen the region without affecting any other part of the mesh.

The coarsening region shown in Figure 10 extends from the top to the bottom of the mesh. Suppose the coarsening region is modified so that it only extends a few layers from the top of the mesh, as shown in Figure 11. In this case, the column collapse operation can be used twice to produce a sheet that is contained entirely within the coarsening region. However, as seen in the figure, the first collapse operation is performed on a column which extends beyond the boundaries of the region. Collapsing this column modifies mesh topology and density in areas where coarsening is not desired. This shows that entirely localized coarsening cannot always be accomplished with the column collapse and sheet extraction operations alone.

### 3.2 Entirely Localized Coarsening

The previous examples demonstrate that entirely localized coarsening requires all operations to take place within the boundaries of the selected coarsening region. Referring to Figure 11, it can be seen that the second collapse operation was performed on a column contained within the coarsening region. Collapsing this column produced a sheet contained within the region without affecting any other part of the mesh. Of course, the formation of this column was accomplished through a previous collapse operation that did affect areas outside the coarsening region. Therefore, a critical aspect of entirely localized coarsening is the creation of local columns. Such columns must be formed in the coarsening region without affecting areas outside the region.

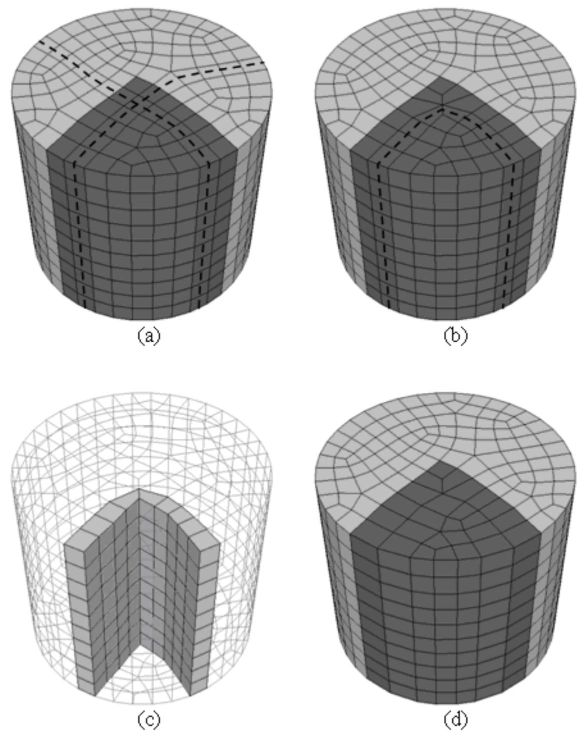


**Fig. 10.** Localized coarsening: (a) Two intersecting sheets pass through a region selected for coarsening. (b) The column defining the intersection of the two sheets in (a) is collapsed to produce a sheet contained entirely within the coarsening region. (c) The sheet that will be extracted. (d) When the sheet in (c) is extracted, mesh density is only decreased within the defined coarsening region.

One way to create local columns without affecting areas outside the coarsening region is through pillowing. As illustrated in Section 2.2, pillowing is a form of refinement because it increases mesh density in the vicinity of the shrink set. For this reason, pillowing is not an obvious solution for coarsening. However, due to the topology constraints that exist in a conforming all-hexahedral mesh, adding elements appears to be a necessary step when coarsening some regions.

Figure 12 shows how pillowing can be used to produce entirely localized coarsening. By pillowing the coarsening region, a sheet is inserted around the region. This sheet intersects other sheets that pass through the coarsening region and provides columns which follow the boundary of the region. Such columns can be collapsed to form sheets contained within the coarsening region without modifying mesh topology or density in areas where coarsening is not desired. These sheets can then be extracted to locally coarsen the region. It should be noted that many of the elements added through pillowing are

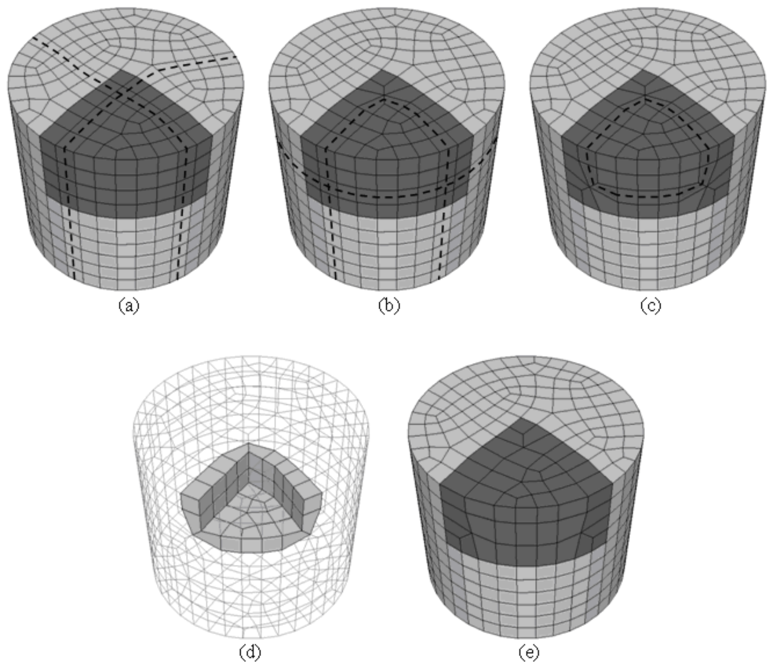


**Fig. 11.** Semi-localized coarsening: (a) Two intersecting sheets pass through a region selected for coarsening. (b) The column defining the intersection of the two sheets in (a) is collapsed. A sheet formed by the collapse and another intersecting sheet are shown. (c) The column defining the intersection of the two sheets in (b) is collapsed to produce a sheet contained entirely within the coarsening region. (d) The sheet that will be extracted. (e) When the sheet in (d) is extracted, mesh density is only decreased within the defined coarsening region.

removed through sheet extraction. Only those elements which are necessary to transition from higher to lower mesh density are left in the mesh. As long as the number of elements removed through sheet extraction is greater than the number of elements added through pillowing, the final mesh density in the coarsening region will be lower than the initial mesh density.

### 3.3 Automated Coarsening Algorithm

For a given region, the process of pillowing, column collapsing, and sheet extraction can be repeated multiple times to achieve various levels of coarsening. A simple algorithm has been developed to automate this process for an arbitrary region and level of coarsening. The overall structure of the algorithm is briefly described by the following steps.

1. A coarsening region is defined and a target mesh density for that region is determined based on input given by a user.

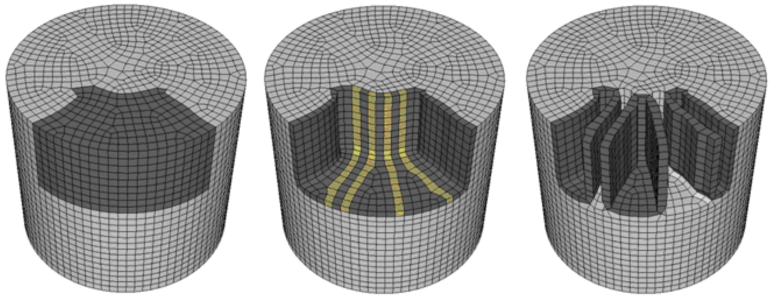


**Fig. 12.** Entirely localized coarsening: (a) A coarsening region is defined. (b) The sheet that forms when the coarsening region is pillowed. This sheet provides columns which follow the boundary of the region. (c) Collapsing the columns in (b) produces sheets contained entirely within the coarsening region.

2. Every sheet that passes through the coarsening region is found. Sheets contained entirely within the coarsening region are distinguished from those that extend beyond the region.
3. Due to a variety of geometry and mesh topology constraints, each sheet is examined to see if it will facilitate valid collapses and extractions during the coarsening process. Sheets that are unable to facilitate valid collapses and extractions are ignored from this point on.
4. For each acceptable sheet, a shape quality metric [22] is used to estimate how the quality of the mesh will be affected if that sheet, or the portion of that sheet contained in the coarsening region, is extracted. Sheets that will potentially produce a higher mesh quality are given higher priority.
5. If there are any sheets contained entirely within the coarsening region, then valid combinations of those sheets are analyzed. The combination that, when extracted, will produce a mesh density that is closest to the target mesh density without over-coarsening is saved. If no acceptable combination is found, the algorithm moves to step 6. Otherwise, steps 6 through 8 are skipped because no other operations are needed before sheet extraction.
6. If there are any sheets that extend beyond the coarsening region, then valid combinations of those sheets are analyzed. For each combination, two coarsening options are possible, as shown in Figure 13. These two coarsening options are distinguished by which direction the columns are collapsed. The combination that will produce a mesh density that is closest to the target mesh density without over-coarsening is saved. If no acceptable combination is found, steps 7 through 9 are skipped.
7. A sheet is inserted around the boundary of the coarsening region through pillowing.
8. Columns in the pillow sheet are collapsed in directions which were previously determined when the best sheet combination was saved. These

collapses form sheets which are contained entirely within the coarsening region.

9. Sheets contained entirely within the coarsening region are extracted.
10. Steps 2 through 9 are repeated until the target mesh density is achieved (within a certain tolerance) or no more valid sheet combinations are found.
11. If coarsening took place, the remaining elements in the region are smoothed to improve mesh quality [23].

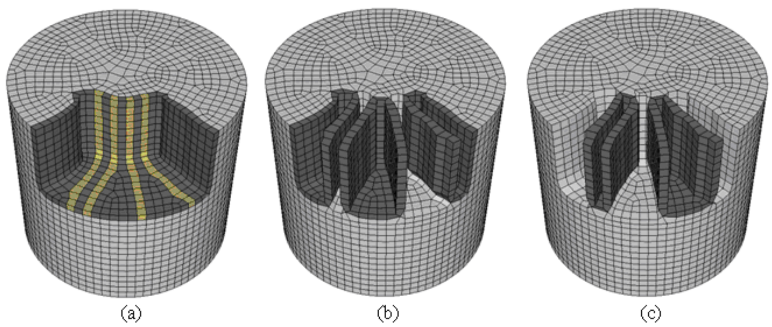


**Fig. 13.** Two coarsening options: (a) Columns selected for collapsing. (b) The sheets that will form if the columns are collapsed one way. (c) The sheets that will form if the columns are collapsed the other way.

## 4 Examples

The following three examples show some results of the automated coarsening algorithm described in Section 3.3. In each example, the goal was to remove 25, 50, and 75 percent of the elements in the region selected for coarsening, while maintaining acceptable element quality. Quality was measured using the scaled Jacobian [24].

The first example was performed on a structured mesh of a cube, as shown in Figure 14. The second example was performed on an unstructured multiple-source to single-target swept mesh of a mechanical part, as shown in Figure 15. The final example was performed on an unstructured mesh of a human head generated with an octree based, sheet insertion algorithm [25], as shown in Figure 16. For both the mechanical part and human head models, refinement was performed prior to coarsening to create a higher starting mesh density.

Tables 1, 2, and 3 provide element removal, element quality, and coarsening time results for each model. In almost every case, acceptable element quality was maintained and a density that very nearly reflects the target mesh density was achieved.

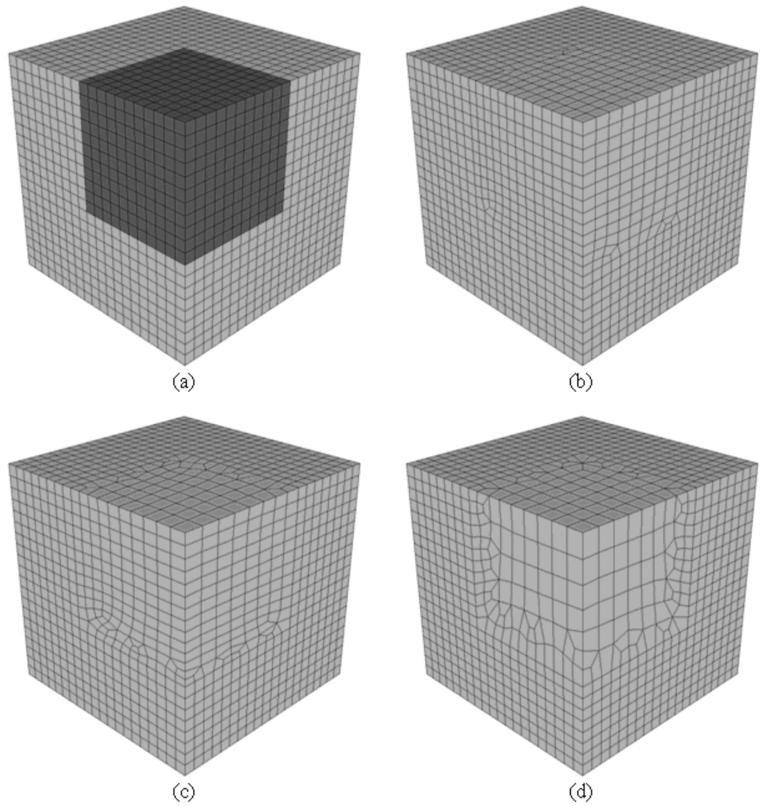


**Fig. 14.** Structured cube example: (a) Original mesh with coarsening region defined. (b) 25 percent coarsening. (c) 50 percent coarsening. (d) 75 percent coarsening.

**Table 1.** Coarsening Results for Cube Model

| Target Percent Removal | Elements in Region | Actual Percent Removal | Min. Scaled Jacobian | Coarsening Time (sec) |
|---|---|---|---|---|
| 0 | 1331 | – | 1.00 | – |
| 25 | 1056 | 20.7 | 0.47 | 0.7 |
| 50 | 684 | 48.6 | 0.41 | 0.9 |
| 75 | 355 | 73.3 | 0.34 | 1.1 |

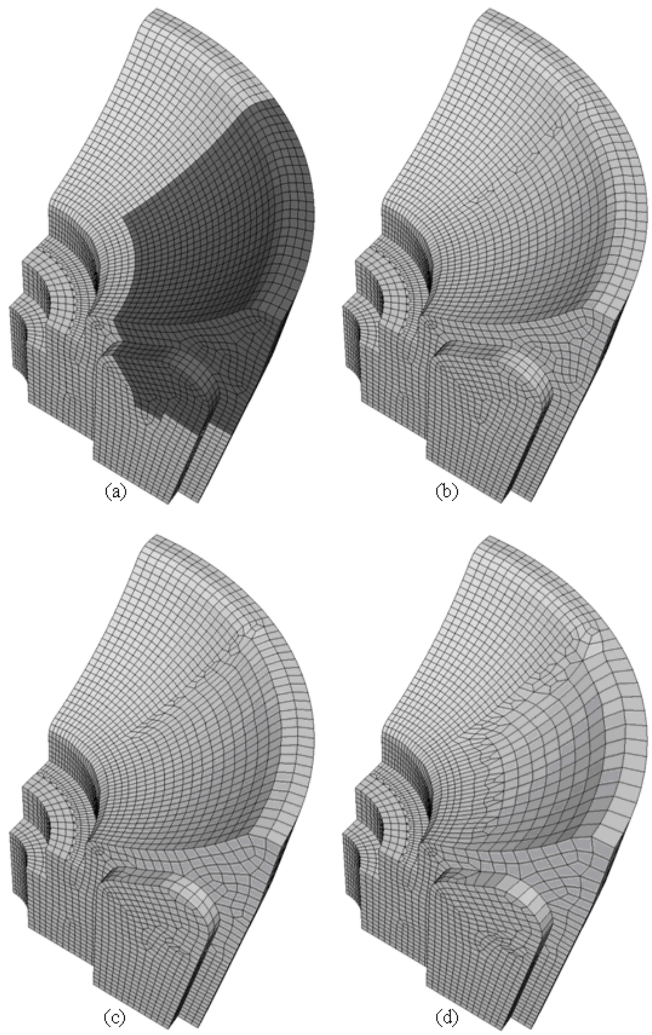


**Fig. 15.** Unstructured mechanical part example: (a) Original mesh with coarsening region defined. (b) 25 percent coarsening. (c) 50 percent coarsening. (d) 75 percent coarsening.

**Table 2.** Coarsening Results for Mechanical Part Model

| Target Percent Removal | Elements in Region | Actual Percent Removal | Min. Scaled Jacobian | Coarsening Time (sec) |
|---|---|---|---|---|
| 0 | 7641 | – | 0.77 | – |
| 25 | 5807 | 24.0 | 0.59 | 5.3 |
| 50 | 4057 | 46.9 | 0.32 | 9.6 |
| 75 | 2205 | 71.1 | 0.22 | 12.5 |

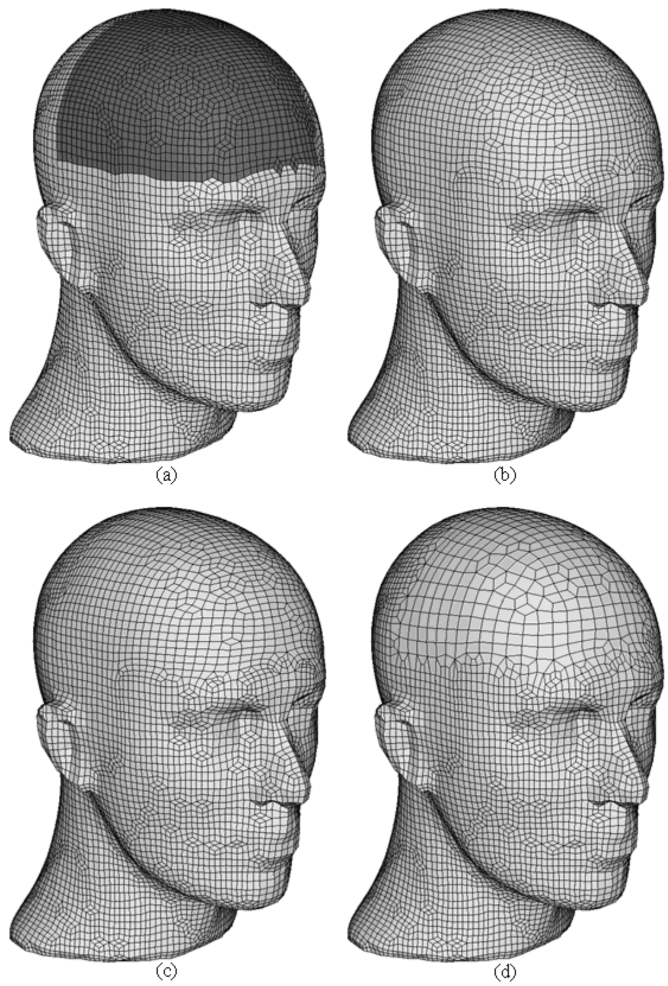


**Fig. 16.** Unstructured human head example: (a) Original mesh with coarsening region defined. (b) 25 percent coarsening. (c) 50 percent coarsening. (d) 75 percent coarsening.

**Table 3.** Coarsening Results for Human Head Model

| Target Percent Removal | Elements in Region | Actual Percent Removal | Min. Scaled Jacobian | Coarsening Time (sec) |
|---|---|---|---|---|
| 0 | 10080 | – | 0.48 | – |
| 25 | 7953 | 21.1 | 0.29 | 13.0 |
| 50 | 5129 | 49.1 | 0.17 | 17.9 |
| 75 | 2615 | 74.1 | 0.22 | 22.5 |

## 5 Future Work

The automated coarsening algorithm described in Section 3.3 takes advantage of sheets already existing entirely within the coarsening region which can be extracted without any previous operations. However, it does not take advantage of columns already existing entirely within the coarsening region which can be collapsed without any previous operations. Modifying the algorithm to take advantage of such columns would improve the efficiency and effectiveness of the coarsening process in certain situations.

While the automated coarsening algorithm guarantees a topologically conforming mesh, it does not guarantee that the final quality of the mesh will be acceptable. Further research is needed to ensure that hexahedral coarsening does not degrade mesh quality below an acceptable threshold. This might be accomplished through more sophisticated methods which prevent poor quality elements from forming, or through cleanup operations which fix bad elements without significantly affecting mesh density. Many effective methods to cleanup a quadrilateral mesh have recently been developed. It is hoped that further research will lead to similar methods for a hexahedral mesh.

The coarsening method presented in this paper has been shown to work on unstructured meshes. However, even though these meshes are considered to be unstructured, they are usually structured in one dimension. Little work has been done to test this method on completely unstructured meshes. In theory, the method should work for any hexahedral mesh. However, it is likely that some meshes cannot be coarsened without degrading element quality below an acceptable level. Further work is needed to determine the limits of this coarsening method.

## 6 Conclusion

By utilizing sheet and column operations such as pillowing, column collapsing, and sheet extraction, entirely localized coarsening can be achieved in conforming all-hexahedral meshes. This method of coarsening works on both structured and unstructured meshes and is not based on undoing previous refinement. Although not fully developed, automation of this hexahedral coarsening method has already shown promising results. However, further work is needed to ensure acceptable element quality and to improve the efficiency of the overall process.

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