On the Existence of a Perfect Matching for 4-Regular Graphs Derived from Quadrilateral Meshes

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In 1891, Peterson [6] proved that every 3-regular bridge-less graph has a perfect matching. It is well known that the dual of a triangular mesh on a compact manifold is a 3-regular graph. M. Gopi and D. Eppstein [4] use Peterson's theorem to solve the problem of constructing strips of triangles from triangular meshes on a compact manifold. P. Diaz-Gutierrez and M. Gopi [3] elaborate on the creation of strips of quadrilaterals when a perfect matching exists. In general, not all 4-regular graphs have a perfect matching. An example planar, 4-regular graph without a perfect matching is given in this paper. However, in this paper, it is shown that the dual of a quadrilateral mesh on a 2-dimensional compact manifold with an even number of quadrilaterals (which is a 4-regular graph) always has a perfect matching.

1 Introduction

In 1891, Peterson [6] showed that every 3-regular graph has a perfect matching. Recent attention has highlighted this result, and M. Gopi and D. Eppstein [4] use Peterson's theorem to solve the problem of constructing strips of triangles from triangular meshes on a compact manifold. Additionally, P. Diaz-Gutierrez and M. Gopi [3] elaborate on the creation of strips of quadrilaterals when a perfect matching exists. They conjecture that every quadrilateral mesh with an even number of elements has a perfect matching, but do not supply a proof.

In this paper, we will prove that a 4-regular graph that is the dual of a quadrilateral mesh on a 2-dimensional compact manifold with an even number of quadrilaterals has a perfect match. This proof is helpful in constructing strips of quadrilaterals from quadrilateral meshes when displaying meshes graphically [3]. This proof is also necessary for a recent result in constructing hexahedral meshes in volumes with constrained quadrilateral meshes on the volume boundaries [1, 2].

Section 2 will introduce the basic definitions in Graph Theory that will be used throughout our discussion together with the existence theorem needed to prove the existence of a perfect matching for quadrilateral meshes.

2 Graph Theoretical Definitions

Before beginning our discussion of our main results, we would like to define and establish basic terms and results that will be used throughout our discussion.

Def. 1.1 *A graph* $\boldsymbol{G} = \langle \boldsymbol{V}, \boldsymbol{E}, \boldsymbol{f} \rangle$ *consists of a finite set* $\boldsymbol{V}$ *of vertices, with a set of edges* $\boldsymbol{E}$*, and a function* $\boldsymbol{f}$ *that maps every edge of* $\boldsymbol{E}$ *to an unordered pair of elements in* $\boldsymbol{G}$*.*

Def. 1.2 *The degree of a vertex is the number of edges that have that vertex as an endpoint.*

Def. 1.3 *A graph G is* ***d-regular*** *if all its vertices have degree* ***d****.*

Def. 1.4 *A* ***one-factor*** *of a graph* $\boldsymbol{G}$*, also known as a* ***perfect matching****, is a sub-graph of* $\boldsymbol{G}$ *that is* ***1-regular*** *and contains every vertex of* $\boldsymbol{G}$*.*

Def. 1.5 *For any subset* $\boldsymbol{S}$ *of* $\boldsymbol{G}$*,* $Z_G(\boldsymbol{S})$ *is the number of edges in* $\boldsymbol{G}$ *that contain precisely one vertex in* $\boldsymbol{S}$*.*

Def. 1.6 *A graph* $\boldsymbol{G}$ *is connected if edges in* $\boldsymbol{G}$ *that contain precisely one vertex is not empty for every formal subset* $\boldsymbol{S}$ *of* $\boldsymbol{V}$*.*

Equivalently, one can state that a set **G** is connected if $Z_G(\mathbf{S}) > 0$ for every formal subset **S** of **G**.

Tutte's Theorem [7] is a fundamental result in perfect matching theory. It provides necessary and sufficient conditions for the existence of a perfect match

Theorem 1.0 (Tutte [7])
A graph $\boldsymbol{G}$ *has a perfect match if and only if, for every subset* $\boldsymbol{S}$ *of the vertices of the graph* $\boldsymbol{G}$*,* ***if the number of connected components with an odd number of vertices in G-S is less than the number of vertices in S.***

In general, not all 4-regular graphs with an even number of vertices have a perfect match (an example 4-regular graph without a perfect matching is illustrated in Figure 1). In this example, **S** has two vertices. **G-S** has 4 connected components; three of them have an odd number of vertices violating the necessary condition in Theorem 1.0. Additionally, the graph shown in Figure 1 is a 4-regular, planar graph that is not dual to a quadrilateral mesh, because, as we will see in theorem 3.1 later in this paper, every quadrilateral mesh on a compact manifold has a perfect matching.

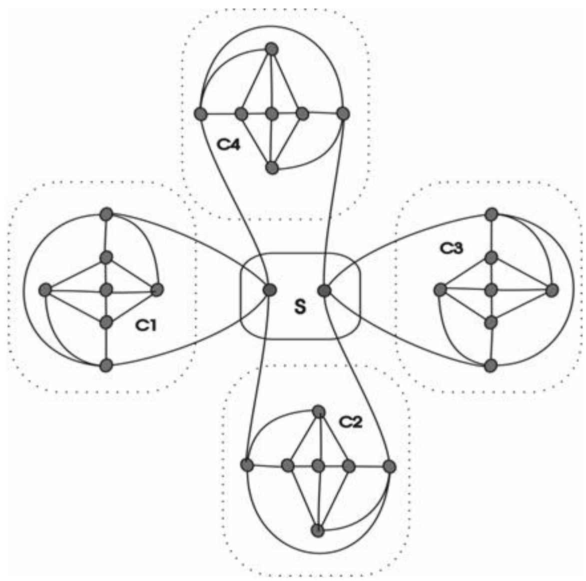

Fig. 1. A planar 4-regular graph with an even number of vertices which does not have a perfect matching, and is not dual to a quadrilateral mesh.

Next we introduce the main theorem that will be used to prove the existence of a perfect matching. The theorem is not as general as Tutte's theorem [7], but does provide a more powerful result on the existence of one-factors for the case of regular graphs.

Theorem 1.1 ([8])
If $\boldsymbol{G}$ *is a regular graph of degree* $\mathbf{d}$ *with an even number of vertices, and if* $Z_G(\boldsymbol{S}) \geq \boldsymbol{d\text{-}1}$ *for every odd-order formal subset of* $\boldsymbol{S}$ *of* $\boldsymbol{V}$ *then each edge of* $\boldsymbol{G}$ *belongs to some one-factor.*

3 Mesh Definitions and Assumptions

In 2-dimensional, finite-element analysis applications, surfaces are partitioned into polygonal domains. The most commonly used types of polygonal shapes are triangles and quadrilaterals. In these applications, the partition of space will need to meet strict conditions. Mitchell [5] provides a formal definition for a purely quadrilateral mesh.

Def. 2.1 *A quadrilateral mesh **M** is a geometric cell complex [5] composed of 0-Dimensional nodes **N**, 1-dimensional edges **E**, and 2-dimensional quadrilaterals **Q** such that:*

A. *Each node is contained by at least two edges.*
B. *Each edge contains two distinct nodes.*
C. *If two edges contain the same nodes, the edges are identical.*
D. *Each quadrilateral is bounded by a cycle of four distinct edges.*
E. *Two nodes have at most one edge between them.*
F. *Two quadrilaterals share at most one edge [1].*
G. *Every edge is contained by no more than two quadrilaterals, and not less than one.*
H. *If two quadrilaterals share four nodes, they are identical.*

When dealing with a quadrilateral mesh on a compact 2-manifold in 3, the following requirement is added:

I. *Every edge is contained by exactly two distinct quadrilaterals.*

Conditions **(A)** through **(I)** in definition 2.1 will be referred from this point on as **(A)** through **(I)** for convenience.

Def. 2.2 *The boundary of any quadrilateral mesh is the collection of edges that are contained by only one quadrilateral.*

Def. 2.3 *Internal edges in a mesh are the edges that are not contained by the set of boundary edges.*

We now make and prove several propositions.

Prop. 2.1 *Every node in the boundary of a mesh is connected to an even number of boundary edges.*

Proof - Let **n** be a node in the mesh **M**. Let **i** be the number of interior edges that contain **n**, **b** be the number of boundary edges that contain **n**, and **q** the number of quadrilaterals that contain **n**. Any quadrilateral with the node **n** has two edges containing **n**; from this observation and condition **(G)** above we can conclude

$$2 * q = 2 * i + b$$

Thus, **b** must be even.

Prop. 2.2 *The boundary of any quadrilateral mesh will have an even number of edges.*

Proof - Similar to the one above, let **b** be the number of boundary edges, let **i** be the number of internal edges, and **q** the total number of quadrilaterals. Then,

$$4 * q = 2 * i + b$$

Thus, **b** must be even.

Prop. 2.3 *The number of edges in the boundary of a quadrilateral mesh is greater than or equal to 4, if it is not empty.*

Proof - By **Prop. 2.2**, the number of edges is even. Suppose the boundary has two edges, then by **Prop. 2.1**, the nodes of the boundary are contained by an even number of boundary edges. Hence, the boundary containing two edges must contain two nodes only. But, by condition **(C)** above, the two edges would be the same, which is a contradiction. Hence the number of edges must be greater than two, but, being even, must be greater than or equal to four. This concludes the proof.

Def. 2.3 *The dual of a mesh* ***M****, is the graph* $\boldsymbol{M^* = \langle Q,\ E,\ f\ \rangle}$ *where* ***Q*** *is the set of quadrilaterals,* ***E*** *is the set of internal edges of* ***M****, and* ***f*** *is the function that maps each edge in* ***E*** *to the set of two unique quadrilaterals that contain it.*

When condition **(I)** holds, dual of the graph is a 4-regular graph. In this case, the set of boundary edges of the quadrilaterals **Q** is empty, because every edge is shared by two quadrilaterals.

4 Existence of a Perfect Match

We now show that a quadrilateral mesh on a closed-compact manifold has a perfect matching.

Def. 3.1 *A mesh* ***M*** *is said to be connected if the graph* ***M**** *is connected.*
Prop. 3.1 *Let* ***M*** *be a connected mesh such that condition* ***(I)*** *is satisfied (i.e. a quadrilateral mesh on a closed-compact manifold). For every formal subset* ***S*** *of* ***Q****,* Z_{M^*} ***(S)*** $\geq$ ***4****.*

Proof - If **C** is a formal subset of **Q**, **Q-C** is not empty. The graph **M*** is connected because **M** is connected by **Def. 3.1** therefore, the set **D** of edges that have precisely one endpoint in **C** is not empty. By definition,

$$Z_{M^*}(S) = |D|$$

Every edge of **D** is in the boundary of **C**, because only one element of **C** contains it. Every edge **e** in the boundary of **C** must be contained by exactly two quadrilaterals by **(I)**. Being a boundary edge of **C**, one of the quadrilaterals of **C** contains the edge **e**, and the other quadrilateral containing **e** must be in **Q-C**. Therefore the boundary of **C** is **D**. By **Prop. 2.3**, this set must be greater than or equal than 4. This concludes the proof.

Next, we prove our claim.

Theorem 3.1 *If a mesh **M** is such that **(I)** is satisfied, and **Q** has an even number of quadrilaterals, every edge of **E** belongs to a perfect matching of **M***.*

Proof - By **Prop. 3.1**, Z_{M^*}(S) $\geq$ 4. Hence, by **Theorem 1.1**, every edge of **M*** belongs to a perfect match.

5 Conclusions

By **Theorem 3.1**, it can be concluded that Figure 1 represents a 4-regular planar graph that is not the dual of a quadrilateral mesh. Thus, one may ask the question of when is a 4-regular graph the dual of a quadrilateral mesh. Although the question is geometric in nature, one can imagine that the conditions for the existence of such a mesh must be purely topological.

One may revise the definition of the mesh **M**, and replace the notion of a quadrilateral for the notion of a n-sided polygon in conditions **(A)** through **(I)**. Such that conditions **(D)** and **(H)** can be replaced by

D. An n-sided polygon is bounded by a cycle of n-edges.
H. If two n-sided polygons share **n** nodes, they are identical.

Thus, the following version of **Prop. 2.1** is valid:

Prop. 4.1 *Every node in the boundary of a mesh of n-sided polygons is connected to an even number of boundary edges.*

The proof of this proposition follows the one in **Prop.2.1**.

In the special case of even **n**, the following versions of **Prop. 2.2** and **Prop. 3.1** hold.

Prop. 4.2 *The boundary of any mesh of n-sided polygons will have an even number of edges if **n** is even.*

The proof of this proposition follows the one given **Prop. 2.2**, with the formula

$$n * q = 2 * i + b$$

The result follows from the fact that **n** is even.

Prop. 4.3 *Let **M** be a connected mesh such that condition **(I)** is satisfied (i.e. a mesh of n-sided polygon on a closed-compact manifold). For every formal subset **S** of **Q**, Z_{M^*} **(S)** $\geq$ 4 if **n** is even.*

The proof is very similar to that of **Prop 3.1**.

Proposition 4.3 states that the dual of meshes of n-sided polygons are n-regular, 4-edge connected graphs if **n** is even.

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