SOLVING ELLIPTIC FINITE ELEMENT SYSTEMS IN NEAR-LINEAR TIME WITH SUPPORT PRECONDITIONERS\textsuperscript{*}

ERIK G. BOMAN\textsuperscript{1}, BRUCE HENDRICKSON\textsuperscript{2}, AND STEPHEN VAVASIS\textsuperscript{3}

Abstract. We consider linear systems arising from the use of the finite element method for solving a certain class of linear elliptic problems. Our main result is that these linear systems, which are symmetric and positive semidefinite, are well approximated by symmetric diagonally dominant matrices. Our framework for defining matrix approximation is support theory. Significant graph theoretic work has already been developed in the support framework for preconditioners in the diagonally dominant case, and in particular it is known that such systems can be solved with iterative methods in nearly linear time. Thus, our approximation result implies that these graph theoretic techniques can also solve a class of finite element problems in nearly linear time. We show that the quality of our approximation, which controls the number of iterations in the preconditioned iterative solver, depends primarily on a mesh quality measure but not on the problem size or shape of the domain.

1. Introduction. Finite element discretizations of elliptic partial differential equations (PDEs) give rise to large sparse linear systems of equations. A topic of great interest is preconditioners for iterative solution of such systems. We focus on scalar boundary value problems of the form $\nabla \cdot (\theta \nabla u) = -f$, in which $\theta$ is a scalar conductivity field. See (3.1) below for a more detailed statement of the PDE under consideration. Such PDEs arise in a variety of physical applications listed in Section 3. We prove that the stiffness matrix $K$ of this PDE, which is defined precisely by (3.7) below, can be well approximated by diagonally dominant linear systems. Since significant theory has been developed for diagonally dominant matrices, our result shows that the same theory extends to this class of stiffness matrices. In particular, our approximation result means that the system $Kx = f$ arising from the finite element method (FEM) can be solved in nearly linear time by preconditioned iterative methods.

Our analysis uses the support theory framework described in [5] for analyzing condition numbers and (generalized) eigenvalues for preconditioned systems. Our analysis is as follows. In Sections 5–7, we state and prove our theorem that $K$ may be factored as $K = A^T \bar{D}^{1/2} HD^{1/2} A$. A preliminary general-purpose result in Section 2 shows that a matrix of this form can be approximated by $K = A^T DA$ (a diagonally dominant matrix), with the quality of approximation depending on $\kappa(H)$. The analysis of $\kappa(H)$ in Section 7 establishes that this quantity depends on the space dimension $d$ and degree $p$ of the FEM, the quadrature rule, and various mesh quality measures but does not depend on the number of nodes or elements or on the shape of the domain or size of elements. It also does not depend on the conductivity field $\theta$ under certain assumptions to be made below.

The idea of approximating FEM systems by diagonally dominant matrices is not

\textsuperscript{*}The work of the first two authors was funded by the Applied Mathematical Sciences program, U.S. Department of Energy, Office of Energy Research and performed at Sandia National Labs, a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the U.S. Dept. of Energy’s National Nuclear Security Administration under contract number DE-AC-94AL85000. The third author was supported in part by NSF Grant CCF-0085969.

\textsuperscript{1}Sandia National Labs, Albuquerque, NM 87185-1318, USA, egboman@sandia.gov.

\textsuperscript{2}Sandia National Labs, Albuquerque, NM 87185-1318, USA, bah@cs.sandia.gov.

\textsuperscript{3}Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON N2L 2W2, Canada, vavasis@math.uwaterloo.ca. Corresponding author. Part of this work was done while the author was in the Department of Computer Science of Cornell University.
new; see for example Gustafsson [13]. In fact, our approach is similar to Gustafsson’s
in that we approximate each element matrix by a diagonally dominant matrix. In
contrast to [13], we are able to rigorously prove bounds on the spectral properties of
our approximation (and thus also the preconditioner).

An approach in [3, §7.3] proposes to precondition general non-diagonally dominant
FEM problems with a diagonally dominant preconditioner obtained by using a lower
order method. Again, our result is a departure from theirs because we get precise and
rigorous bounds on the quality of the approximation.

In Section 10, we specialize the theory to four common cases of the finite element
method and report on computational tests for these cases.

2. Support Theory. Let $A, B$ be symmetric positive semidefinite (SPSD) ma-
trices of the same size, and let $N(A)$ and $N(B)$ denote their null spaces. We define
the support number of $A$ with respect to $B$ to be

$$
\sigma(A, B) = \sup_{x \in \mathbb{R}^n - N(B)} \frac{x^T A x}{x^T B x}.
$$

(2.1)

Note that this quantity is finite only if $N(B) \subset N(A)$. We follow the convention
established in the previous literature of using $\sigma(\bullet, \bullet)$ to denote support numbers.
Unfortunately, $\sigma$ is also commonly used to denote singular values. In this paper, $\sigma$
with one argument is a singular value, and $\sigma$ with two arguments is a support number.

The results in this section can be partially generalized to indefinite symmetric
matrices. This generalization requires a more general definition of support number
than (2.1) and requires the use of the Symmetric Product Support Theorem from [5].
Since this paper focuses on the positive semidefinite case, we omit this generalization.

If $A$ and $B$ are symmetric positive definite (SPD), then $\sigma(A, B) = \lambda_{\text{max}}(A, B)$, the
largest generalized eigenvalue. When $B$ is a preconditioner for $A$ in the preconditioned
conjugate gradient iterative method [10], the condition number of the preconditioned
system is given by $\sigma(A, B) \sigma(B, A)$, which we denote as $\kappa(A, B)$. When $A, B$ are SPD,
then $\kappa(A, B) = \kappa(B^{-1} A)$ where $\kappa$ denotes the standard spectral condition number.

A principal goal of this paper is to propose the construction of a symmetric
diagonally dominant matrix $\bar{K}$ that approximates the finite element stiffness matrix $K$
in the sense that $\kappa(K, \bar{K})$ is not too large. The following two lemmas and subsequent
theorem define our framework for defining $\bar{K}$ and bounding $\kappa(K, \bar{K})$.

Lemma 2.1. Suppose $V \in \mathbb{R}^{n \times p}$, and suppose $H \in \mathbb{R}^{p \times p}$ is SPD. Then

$$
\sigma(V H V^T, V V^T) \leq \lambda_{\text{max}}(H),
$$

where $\lambda_{\text{max}}$ denotes the largest eigenvalue.

Proof.

$$
\sigma(V H V^T, V V^T) = \sup_{x \in \mathbb{R}^n - N(V^T)} \frac{x^T V H V^T x}{x^T V V^T x} = \sup_{y \in R(V^T)} \frac{y^T H y}{y^T y} \leq \lambda_{\text{max}}(H).
$$

The second line follows from the first by substituting $y = V^T x$ and the third from the
Courant-Fischer minimax theorem [10, §8.1.1]. Here, $R(V^T)$ denotes the range-space
of $V^T$. \[}
Lemma 2.2. Suppose $V \in \mathbb{R}^{n \times p}$, and suppose $H \in \mathbb{R}^{p \times p}$ is SPD. Then
\[
\sigma(VV^T, VHV^T) \leq 1/\lambda_{\min}(H),
\]
where $\lambda_{\min}$ denotes the smallest eigenvalue.

Proof. Using the same idea in the previous proof leads to a supremum over the Rayleigh quotient $y^T y / (y^T H y)$, which is bounded above by $1/\lambda_{\min}(H)$. \qed

Combining these two lemmas yields a result for the condition number.

Theorem 2.3. Suppose $V \in \mathbb{R}^{n \times p}$, and suppose $H \in \mathbb{R}^{p \times p}$ is SPD. Then
\[
\kappa(VV^T, VHV^T) \leq \kappa(H).
\]

Proof. The result follows from Lemma 2.1 and 2.2 and by using the fact $\kappa(H) = \lambda_{\max}(H)/\lambda_{\min}(H)$ when $H$ is SPD. \qed

The way we will apply this theorem is to let $V = A^T \bar{D}^{1/2}$, where $A$ is the node-arc incidence matrix of a graph and $\bar{D}$ is a diagonal weight matrix. Then $K = VV^T = A^T \bar{D} A$ is diagonally dominant while $K = VHV^T = A^T \bar{D}^{1/2} H \bar{D}^{1/2} A$ is not (in general). Therefore, we now have a tool to approximate non-diagonally-dominant matrices.

3. Finite element analysis. In this section we provide a brief summary of isoparametric finite element approximation as well as an introduction to notation that is crucial for our main theorem. The material in this section is standard [15] in textbooks, except that our description herein uses notation for indexing that is more detailed than usual. This section concludes with a summary of our notation.

The class of problems under consideration consists of finite-element discretizations of the following second-order elliptic boundary value problem. Find $u : \Omega \to \mathbb{R}$ satisfying
\[
\begin{align*}
\nabla \cdot (\theta \nabla u) &= -f & \text{on } \Omega, \\
u &= u_0 & \text{on } \Gamma_1, \\
\theta \partial u / \partial n &= g & \text{on } \Gamma_2.
\end{align*}
\]

Here, $\Omega$ is a bounded open subset of $\mathbb{R}^d$ (typically $d = 2$ or $d = 3$), $\Gamma_1$ and $\Gamma_2$ form a partition of $\partial \Omega$, $\theta$ is a given scalar field on $\Omega$ that is positive-valued everywhere and is sometimes called the conductivity, $f : \Omega \to \mathbb{R}$ is a given function called the forcing function, $u_0$ is a given function called the Dirichlet boundary condition and $g$ is another given function called the Neumann boundary condition.

This problem has applications to many problems in mathematical physics. For example, $u$ can represent voltage in a conducting medium, in which case $\theta$ stands for electrical conductivity. The two types of boundary conditions stand for, respectively, a boundary point held at fixed voltage or a boundary point electrically insulated (or with prespecified nonzero current). Another application is thermodynamics, in which $u$ stands for temperature of the body, $\theta$ for thermal conductivity, $\Gamma_1$ for a boundary held at fixed temperature, and $\Gamma_2$ for a boundary insulated (or with prespecified heat flow). Problem (3.1) is also used to model membrane deflection and gravity. It arises as a subproblem in fluid flow modeling.

For clarity, we keep track of our assumptions by explicitly numbering them. One assumption has already been made:

Assumption 1. For all $x \in \Omega$, $\theta(x) > 0$. 

3
Without this assumption, the problem may be ill posed.

The first step in the isoparametric finite element method is to produce a mesh of this domain. For the remainder of the paper, we assume isoparametric elements are defined with respect to the usual polynomial basis on a simplicial reference element, although the results can be generalized to other reference elements and basis families.

In more detail, let $T_0$ denote the standard unit $d$-simplex: for $d = 2$, this simplex is the triangle with vertices $(0,0), (1,0), (0,1)$, and for $d = 3$ this simplex is the tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$. Let $p$ denote the polynomial order of the finite element method. In the reference element, position the nodes on the boundary of $\tilde{\Omega}$ and, in three dimensions, $k$ are nonnegative integers whose sum is at most $p$. Let these nodes be enumerated $z_1, \ldots, z_l$.

One generates a mesh of $\Omega$ composed of $m$ elements to be defined via mapping functions. For each element $t = 1, \ldots, m$, there is a mapping function $\phi_t$ that maps $T_0$ to $\mathbb{R}^d$. Let $\nabla \phi_t$ denote the derivative (Jacobian) of $\phi_t$. The following assumption is nearly universal in the literature.

**Assumption 2.** Mapping function $\phi_t : T_0 \rightarrow \mathbb{R}^d$ is one-to-one and onto, and $\det(\nabla \phi_t(z)) > 0$ for all $z \in T_0$, $t = 1, \ldots, m$.

The $t$th element is defined to be $\phi_t(T_0)$ and hence has a shape of a curved simplex. The function $\phi_t$ carries the $l$ reference nodes $z_1, \ldots, z_l$ to $l$ real-space nodes $\phi_t(z_1), \ldots, \phi_t(z_l)$, which are often just called nodes. These may be denoted $\zeta_{l,1}, \ldots, \zeta_{l,l}$. The nodes are chosen so that the nodes on the boundary of the $t$th element coincide with the corresponding nodes on the boundaries of its neighbors.

Restrict $\phi_t$ to be a polynomial of degree $p$, in which case it is uniquely determined by the positions of its real-space nodes. In more detail, let $N_{\mu, \nu} = 1, \ldots, l$, be a real-valued degree-$p$ polynomial function on $T_0$ with the property that

$$N_{\mu}(z_\nu) = \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad \text{for all } \nu = 1, \ldots, l. \tag{3.2}$$

It is not hard to write down an explicit formula for $N_{\mu}$ and to prove that (3.2) uniquely determines $N_{\mu}$ among degree-$p$ polynomials. These functions $N_1, \ldots, N_l$ are called shape functions. For each $t$, $\phi_t$ necessarily has the formula $\phi_t = \zeta_{t,1}N_1 + \cdots + \zeta_{t,l}N_l$.

There are many duplicate entries in the list $\zeta_{1,1}, \ldots, \zeta_{m,l}$ because of common nodes at the boundaries of the elements. Let $w_1, \ldots, w_{m'}$ be a listing of the real-space nodes with all duplicates removed. This renumbering is specified by an index mapping function, the local-to-global numbering map, denoted $LG$ and carrying an index pair $(t, \mu)$ to an index $i \in 1, \ldots, n'$ so that $w_i \equiv \zeta_{t,\mu}$.

Finally, the mesh $T$ is specified by listing the nodes $w_1, \ldots, w_{m'}$ and the local-global mapping defined by $LG$. From this data, one can deduce all of the $\phi_t$’s.

Let $\tilde{\Omega}$ be the union of the elements, which may be slightly different from $\Omega$. This is because at the boundary of $\Omega$, the elements may either protrude a bit outside $\Omega$ or may fail to cover a small part of the boundary. Let the boundary of $\tilde{\Omega}$ be partitioned into $\Gamma_1$ and $\Gamma_2$ in correspondence with the partition $\Gamma_1, \Gamma_2$ of the boundary of $\Omega$. It is necessary to transfer $\theta$ (and several other fields) from $\tilde{\Omega}$ to $\Omega$ and to transfer the boundary conditions to $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$. We omit the details.
The next step in the isoparametric finite element method is to define basis functions \( \pi_1, \ldots, \pi_{n'} \), which are functions from \( \tilde{\Omega} \) to \( \mathbb{R} \) satisfying

\[
\pi_i(w_k) = \begin{cases} 
1 & \text{if } i = k, \\
0 & \text{if } i \neq k
\end{cases}
\text{ for } k = 1, \ldots, n'.
\] (3.3)

We require that \( \pi_i \), when restricted to an element \( t \), must be of the form \( N_{\mu} \circ \phi_t^{-1} \) where \( LG(t, \mu) = i \) or else must be identically 0 if \( i \) does not occur among \( LG(t, 1), \ldots, LG(t, l) \). This requirement uniquely determines the \( \pi_i \)'s. It can be shown that \( \pi_i \) is continuous but fails to be differentiable at inter-element boundaries.

Let the global numbering be chosen so that \( w_1, \ldots, w_n \), the first \( n \) real-space nodes, are the nodes with no Dirichlet boundary condition imposed, and let the final \( n' - n \) nodes have Dirichlet boundary conditions. Now we can define the exact assembled stiffness matrix \( K_{\text{exact}} \) of the finite element method to be the \( n \times n \) matrix whose \((i, j)\) entry is given by

\[
K_{\text{exact}}^{\text{exact}}(i, j) = \int_{\tilde{\Omega}} \nabla \pi_i(x) \cdot \theta(x) \nabla \pi_j(x) \, dx
\] (3.4)

where \( \nabla \) as usual denotes the gradient.

Matrix \( K_{\text{exact}}^{\text{exact}} \) is sparse, symmetric and positive semidefinite. Symmetry is obvious; semidefiniteness follows from a fairly straightforward argument that we omit, and sparsity follows because \( K_{\text{exact}}^{\text{exact}}(i, j) \) is nonzero only if there is an element \( t \) that contains both nodes \( w_i \) and \( w_j \).

Integral (3.4) is difficult to compute directly because evaluating \( \pi_i \) requires evaluation of \( \phi_t^{-1} \). Fortunately, this difficulty is avoided by breaking the integral into a sum over elements then carrying out the integral over the reference domain following a change of variables as follows.

\[
K_{\text{exact}}^{\text{exact}}(i, j) = \sum_{t=1}^{m} \int_{T_0} \nabla_x \pi_i(\phi_t(z)) \cdot \theta(\phi_t(z)) \nabla_x \pi_j(\phi_t(z)) \det(\nabla \phi_t(z)) \, dz,
\] (3.5)

where the notation \( \nabla_x \) means derivative with respect to the coordinates of element \( t \) (as opposed to derivative with respect to \( z \), the coordinates of \( T_0 \)).

The integrand of (3.5) is evaluated using the chain rule for derivatives. Assume \( w_i \) is a node of element \( t \) (else the above integral is 0). Let \( \mu \) be the index such that \( LG(t, \mu) = i \), so that \( \pi_i = N_{\mu} \circ \phi_t^{-1} \) on element \( t \). Then

\[
\nabla_x \pi_i(\phi_t(z)) = \nabla_x N_{\mu}(z) = \nabla \phi_t(z)^{-T} \cdot \nabla_x N_{\mu}(z),
\] (3.6)

and similarly for \( \pi_j \). Here, \( \nabla \phi_t(z)^{-T} \) denotes the transposed inverse of the \( d \times d \) matrix \( \nabla \phi_t(z) \), which exists by Assumption 2. The ‘\( \cdot \)’ notation in the previous formula indicates matrix-vector multiplication. Gradients are regarded as length-\( d \) column vectors.

We assume that the entries of \( K \) are not the exact value of the integral (3.5) but are obtained by a quadrature rule that we now discuss. Let \( r_1, \ldots, r_q \) be points in the interior of the reference element \( T_0 \) called the Gauss points. (As is common practice, we use this terminology even if the quadrature rule is not derived from Gaussian quadrature.) Let \( \omega_1, \ldots, \omega_q \) be corresponding Gauss weights. We denote
this quadrature rule, i.e., the set of ordered pairs \((r_1, \omega_1), \ldots, (r_q, \omega_q)\), by the symbol \(Q\). Then in place of (3.5) we take

\[
K(i, j) = \sum_{t=1}^{m} \sum_{k=1}^{q} \nabla x \pi_i(\phi_t(r_k)) \cdot \theta(\phi_t(r_k)) \nabla x \pi_j(\phi_t(r_k)) \det(\nabla \phi_t(r_k)) \omega_k,
\]

in which \(\nabla x \pi_i(\phi_t(r_k))\) is evaluated by substituting \(z = r_k\) into the right-hand side of (3.6) and similarly for \(\nabla x \pi_j(\phi_t(r_k))\). Symmetry and sparsity of \(K\) follow for the same reason as for \(K^\text{exact}\); positive semidefiniteness will follow from results in the next two sections (under some additional assumptions to be made).

We close this section with a summary of the notation introduced thus far. Integers that define the size of the computation are:

\[d = \text{space dimension of (3.1)},\]
\[p = \text{polynomial order of the finite element method},\]
\[l = \text{number of reference nodes} = \begin{cases} (p+1)(p+2)/2 & \text{if } d = 2, \\ (p+1)(p+2)(p+3)/6 & \text{if } d = 3, \end{cases}\]
\[m = \text{number of elements},\]
\[n = \text{number of non-Dirichlet real-space nodes},\]
\[n' = \text{total number of real-space nodes},\]
\[q = \text{number of Gauss points in the quadrature rule } Q.\]

Sets of points in \(\mathbb{R}^d\) include

\[z_1, \ldots, z_l = \text{reference nodes},\]
\[\zeta_{1,1}, \ldots, \zeta_{m,l} = \text{real-space nodes (local numbering)},\]
\[w_1, \ldots, w_{n'} = \text{real-space nodes (global numbering)}.\]

The quadrature rule \(Q\) is defined by

\[r_1, \ldots, r_q = \text{Gauss points},\]
\[\omega_1, \ldots, \omega_q = \text{Gauss weights}.\]

Important domains are:

\[T_0 = \text{the reference element},\]
\[\Omega = \text{the domain of (3.1)},\]
\[\phi_1(T_0), \ldots, \phi_m(T_0) = \text{the elements},\]
\[\tilde{\Omega} = \text{the approximation to } \Omega \text{ given by } \phi_1(T_0) \cup \cdots \cup \phi_m(T_0).\]

Important functions are:

\[\phi_1, \ldots, \phi_m = \text{element mapping functions } (T_0 \to \tilde{\Omega}),\]
\[N_1, \ldots, N_l = \text{shape functions } (T_0 \to \mathbb{R}),\]
\[\theta = \text{conductivity } (\tilde{\Omega} \to \mathbb{R}),\]
\[LG = \text{local-to-global index mapping } \{1, \ldots, m\} \times \{1, \ldots, l\} \to \{1, \ldots, n'\},\]
\[\pi_1, \ldots, \pi_{n'} = \text{basis functions } (\tilde{\Omega} \to \mathbb{R}).\]
Finally, scalar quantities to be introduced in the next section, but which are included here for completeness, include

\[ m_Q, M_Q = \text{min and max quadrature weights,} \]
\[ \hat{\theta}(T) = \text{intra-element conductivity variation,} \]
\[ \alpha_1, \ldots, \alpha_m = \text{maximum compression in elements,} \]
\[ \beta_1, \ldots, \beta_m = \text{maximum stretch in elements,} \]
\[ \kappa_1(T), \kappa_2(T) = \text{mesh quality measures,} \]
\[ \sigma_{Q,p}, \tau_{Q,p} = \text{max and min singular values of} \ S_{Q,p}. \]

4. **Condition numbers and assumptions.** Our main theorem about matrix approximation, which is Theorem 5.1 below, depends on several scalars associated with the finite element method and the problem at hand that we define in this section. In addition, this section states further assumptions about the problem and method.

Two constants appearing in our bound are \( m_Q \) and \( M_Q \), which we define to be the minimum and maximum weights in the quadrature rule, i.e.,

\[ m_Q = \min(\omega_1, \ldots, \omega_q); \quad M_Q = \max(\omega_1, \ldots, \omega_q). \] (4.1)

**Assumption 3.** The quadrature weights are positive, i.e., \( m_Q > 0 \).

There is some loss of generality with this assumption because a few popular finite element quadrature schemes (but certainly not all) use negative weights [8].

**Assumption 4.** The quadrature scheme is exact for polynomials of degree up to \( 2p - 2 \), i.e., if \( \psi : T \rightarrow \mathbb{R} \) is a polynomial of degree \( 2p - 2 \) or less in the \( d \) coordinates of \( T_0 \), then

\[ \sum_{k=1}^{q} \psi(r_k)\omega_k = \int_{T_0} \psi(z) \, dz. \]

This assumption is quite reasonable since it is usually required anyway for accurate solution by finite element analysis: one wants accurate quadrature of \( \nabla \pi_i \cdot \nabla \pi_j \).

We now define a \( dq \times (l - 1) \) matrix \( S_{Q,p} \) according to the following formula:

\[
S_{Q,p} = \begin{pmatrix}
\nabla N_2(r_1) & \nabla N_3(r_1) & \cdots & \nabla N_l(r_1) \\
\vdots & \vdots & & \vdots \\
\nabla N_2(r_q) & \nabla N_3(r_q) & \cdots & \nabla N_l(r_q)
\end{pmatrix}.
\] (4.2)

Although we introduce this matrix in this section in order to define two associated scalars, the motivation for this definition will be postponed until Section 6. Before defining these scalars, we require the following lemma.

**Lemma 4.1.** Under Assumption 4, matrix \( S_{Q,p} \) has full column rank.

**Proof.** Let \( v \in \mathbb{R}^{l-1} \) be chosen so that \( S_{Q,p}v = 0 \). Define the function \( U : T_0 \rightarrow \mathbb{R} \) according to the formula \( U = v_1N_2 + v_2N_3 + \cdots + v_{l-1}N_l \). Observe that, by definition of \( S_{Q,p} \), the first \( d \) entries of \( S_{Q,p}v \) are precisely \( \nabla U(r_1) \), and the next \( d \) entries are \( \nabla U(r_2) \), etc. Therefore, \( \nabla U \) vanishes identically at \( r_1, \ldots, r_q \). Let \( \psi : T_0 \rightarrow \mathbb{R} \) be defined by \( \psi = \nabla U \cdot \nabla U \). Then \( \psi \) has degree at most \( 2p - 2 \), is a nonnegative function, and also vanishes identically at \( r_1, \ldots, r_q \). By Assumption 4, this means that \( \int_{T_0} \psi = 0 \). But since \( \psi \) is nonnegative-valued, we conclude that \( \psi \equiv 0 \). Therefore, \( \nabla U \equiv 0 \) also, and thus \( U \) must be a constant function on \( T_0 \). But the definition of
\( U \) omits the \( N_1 \) term, and therefore \( U(z_1) = 0 \). Since \( U \) is constant, this means that \( U \equiv 0 \) on all of \( T_0 \), and in particular, \( U(z_2) = \cdots = U(z_l) = 0 \). But \( U(z_2) = v_1 \), \( U(z_3) = v_2 \), \ldots, \( U(z_l) = v_{l-1} \) by construction of \( U \). Therefore, the entries of \( v \) are all zeros. Since \( v \) was an arbitrary vector in \( N(S_{Q,p}) \), this argument proves that this nullspace contains only the \( 0 \) vector, hence \( S_{Q,p} \) has full column rank. \( \square \)

The next constants that appear in our theorem are \( \sigma_{Q,p} \) and \( \tau_{Q,p} \), defined as follows:

\[
\sigma_{Q,p} = \max(S_{Q,p}); \quad \tau_{Q,p} = \min(S_{Q,p}).
\]

These constants depend only on \( p \) and the quadrature scheme. It follows from the lemma that both are positive.

The remaining scalars and assumptions in this section pertain to the mesh. Define for \( t = 1, \ldots, m \),

\[
\alpha_t = \max\{||\nabla \phi_t(r_1)^{-1}||, \ldots, ||\nabla \phi_t(r_q)^{-1}||\},
\]

\[
\beta_t = \max\{||\nabla \phi_t(r_1)||, \ldots, ||\nabla \phi_t(r_q)||\}.
\]

The norms in these equations are matrix 2-norms: \( \|B\| \equiv \sigma_{\text{max}}(B) \). Finally, for the whole mesh, a quality measure is

\[
\kappa_1(T) = \max_{t=1,\ldots, m} \alpha_t \beta_t.
\]

Although the \( \alpha \)'s and \( \beta \)'s are subscripted only by \( t \), it is clear from the definition that they also depend on \( Q \). On the other hand, it is possible to get a \( Q \)-independent definition of these by simply taking the upper bounds similar to (4.4), (4.5) over all \( z \in T_0 \) instead of just \( r_1, \ldots, r_q \). Unfortunately, for higher order elements \( (p > 1) \), there is no simple method to compute \( \max_{z \in T_0} ||\nabla \phi_t(z)^{-1}|| \); a technique for obtaining an upper bound appears in [21].

It should be noted that \( \kappa_1(T) \geq 1 \) since for any \( z \), \( ||\nabla \phi_t(z)^{-1}|| \cdot ||\nabla \phi_t(z)|| \geq 1 \). If all the elements are well-shaped, i.e., not too distorted when compared to the reference element, then \( \kappa_1(T) \) will not be much larger than 1.

The second mesh quality measure is

\[
\kappa_2(T) = \max_{t=1,\ldots, m} \frac{\max_{k=1,\ldots, q} \det(\nabla \phi_t(r_k))}{\min_{k=1,\ldots, q} \det(\nabla \phi_t(r_k))}.
\]

This quantity measures the maximum over elements of the variation in volumetric distortion over the element. This may be regarded as a measure of how much elements depart from linearity (flatness). Measure \( \kappa_2 \) is not completely independent from \( \kappa_1 \) as the following argument shows. It follows from Hadamard’s inequality that

\[
\alpha_t^{-d} \leq \det(\nabla \phi_t(r_k)) \leq \beta_t^d
\]

and therefore

\[
\frac{\max_{k=1,\ldots, q} \det(\nabla \phi_t(r_k))}{\min_{k=1,\ldots, q} \det(\nabla \phi_t(r_k))} \leq (\alpha_t \beta_t)^d,
\]

hence \( \kappa_2(T) \leq \kappa_1(T)^d \). This bound is not likely to be tight in practice (see our computation results in Table 10.3), so we prefer to distinguish the roles of \( \kappa_1 \) and \( \kappa_2 \).
The final two assumptions are qualitative in nature and are meant to indicate cases in which our method is expected to work well.

**Assumption 5.** Mesh quality measures $\kappa_1(T), \kappa_2(T)$ are not too large.

In the case of linear elements ($p = 1$), $\kappa_1(T)$ is proportional to the reciprocal of the minimum angle (in two dimensions) or solid angle (in three dimensions) of the mesh and $\kappa_2(T) \equiv 1$.

Assumption 5 does not imply that the elements are of a uniform size: a uniform rescaling of element $t$ does not affect the product $\alpha_t\beta_t$.

The final constant and assumption pertain to the conductivity field. Define

$$\hat{\theta}(T) = \max_{t=1,\ldots,m} \max_{k=1,\ldots,q} \frac{\theta_t(\phi_t(r_k))}{\min_{k=1,\ldots,q} \theta_t(\phi_t(r_k))}.$$  (4.8)

In other words, $\hat{\theta}(T)$ measures the maximum intra-element variation of the conductivity field $\theta$.

**Assumption 6.** Intra-element conductivity variation $\hat{\theta}(T)$ is not too large.

This assumption implies that if there are huge jumps in $\theta$ in the domain (e.g., because one is modeling a domain composed of two materials with vastly different conductivities), then the mesh boundaries should be aligned with the conductivity jumps. If the mesh boundaries are aligned with the conductivity jumps, then no element will have large variation in $\theta$ among its Gauss points and thus $\hat{\theta}(T)$ will not be large.

Note that, as in the case of $\kappa_1(T)$ and $\kappa_2(T)$, scalar $\hat{\theta}(T)$ also depends on the quadrature rule, but this dependence can be eliminated by overestimating $\hat{\theta}(T)$ as

$$\max_{t=1,\ldots,m} \max_{z \in T_t} \frac{\theta_t(\phi_t(z))}{\min_{z \in T_t} \theta_t(\phi_t(z))}.$$  

5. The matrix approximation. Our main matrix factorization result is summarized by the following theorem whose proof is explained in upcoming sections.

**Theorem 5.1.** Let $K$ be defined by (3.7) above, and let Assumptions 1–4 hold. Then $K$ may be factored as $A^TJ^TDJA$, where

- $A$ is an $(l-1)m \times n$ reduced node-arc incidence matrix of a certain multigraph,
- $J$ is a $dqm \times (l-1)m$ matrix that is well conditioned in the sense that
  
  $$\sigma_{\max}(J) \leq \sigma_{Q,p},$$  
  
  and
  
  $$\sigma_{\min}(J) \geq \tau_{Q,p}/\kappa_1(T).$$  (5.2)

- $D$ is a $dqm \times dqm$ positive definite diagonal matrix.

Further, $J^TJD$ may be refactored as $\hat{D}^{1/2}HD^{1/2}$ where $\hat{D}$ is a $(l-1)m \times (l-1)m$ SPD diagonal matrix and $H$ is a $(l-1)m \times (l-1)m$ SPD matrix whose condition number is bounded as follows:

$$\kappa(H) \leq \hat{\theta}(T)\kappa_1(T)^2\kappa_2(T) \cdot \frac{M_Q\sigma_{Q,p}^2}{m_Q\tau_{Q,p}^2}. $$  (5.3)

This theorem can now be combined with Theorem 2.3 to obtain a good approximation $\hat{K}$ to the stiffness matrix $K$. In particular, we take $\hat{K} = A^TDA$. Note that
this matrix, being a weighted graph laplacian, is symmetric and diagonally dominant. Then in the context of Theorem 2.3, \( V = A^T D^{1/2} \). The theorem thus implies that the condition number of \( \bar{K} \) with respect to \( K \) depends only on the condition number of \( H \) for which we have a good bound (5.3). Then \( \bar{K} \) can be preconditioned using techniques in the previous literature. The complete description of the algorithm is given below in Section 8.

The approximation bound clearly depends on the quadrature rule, mesh quality, and intra-element variation of \( \theta \). It is also informative to make a list of quantities on which \( \kappa(\bar{K}, K) \) does not depend:

- The number of nodes or elements,
- The size of the elements,
- Variation in the size of elements (i.e., gradation of the mesh),
- The shape of \( \Omega \) (except that if \( \Omega \) has a sharp corner, then the mesh will necessarily also have a sharp corner, which forces \( \kappa_1(T) \) to be large),
- The conductivity field \( \theta \) (provided that the mesh respects internal boundaries where large conductivity jumps occur).

6. The first factorization. In this section we develop the first factorization of \( K \) stated in Theorem 5.1. This factorization is related to one proposed in [22]. We state the main result of this section as a lemma.

**Lemma 6.1.** Matrix \( K \) defined by (3.7) can be factored as \( K = A^T S^T R^T D R S A \) where the factors are as follow.

- Define matrix \( A \in \mathbb{R}^{(l-1)m \times n} \) to be a sparse matrix all of whose entries are \(-1, 0 \) or \(1\). In more detail, \( A \) is written in block form

\[
A = \begin{pmatrix}
A_1 \\
\vdots \\
A_m
\end{pmatrix},
\]

where \( A_t \) is \((l-1) \times n\) for each element \( t = 1, \ldots, m \). The columns are indexed 1, \ldots, \( n \) in correspondence with nodes \( w_1, \ldots, w_n \). Row \( \mu - 1 \) of \( A_t \) has a ‘1’ in column \( LG(t, \mu) \) for \( \mu = 2, \ldots, l \) and a ‘-1’ in column \( LG(t, 1) \). Thus, most rows of \( A \) have exactly two nonzero entries. If \( LG(t, \mu) > n \) (i.e., node \( \zeta_{t,\mu} \) lies in the Dirichlet boundary \( \bar{\Gamma}_1 \)), then the ‘1’ entry is omitted. Similarly, if \( LG(t, 1) > n \), then the ‘-1’ entry is omitted. For this reason, a few rows of \( A \) have just one nonzero entry or none at all.

- Define \( S \in \mathbb{R}^{dqm \times \mathbb{R}^{(l-1)m}} \) by

\[
S = \left( \begin{array}{ccc}
S_{Q,p} & \cdots & S_{Q,p}
\end{array} \right)
\]

m times, \( m \)

where \( S_{Q,p} \) was defined by (4.2).

- Define block diagonal \( R \in \mathbb{R}^{dqm \times dqm} \) by

\[
R = \begin{pmatrix}
R_1 \\
\vdots \\
R_m
\end{pmatrix},
\]
where each of $R_1, \ldots, R_m \in \mathbb{R}^{qd \times qd}$ is itself block diagonal and given by

$$R_t = \alpha_t^{-1} \left( \begin{array}{ccc} \nabla \phi_t(r_1)^{-T} & \cdots & \nabla \phi_t(r_q)^{-T} \end{array} \right).$$  \hspace{1cm} (6.3)

The scalar $\alpha_t$ used here was defined by (4.4).

- Finally, $D \in \mathbb{R}^{qdm \times qdm}$ is positive definite diagonal and given by

$$D = \left( \begin{array}{ccc} D_1 & \cdots & \cdots \end{array} \right),$$  \hspace{1cm} (6.4)

where $D_t \in \mathbb{R}^{qd \times qd}$, $t = 1, \ldots, m$, is given by

$$D_t = \alpha_t^2 \left( \begin{array}{ccc} \theta(\phi_t(r_1)) \det(\nabla \phi_t(r_1)) \omega_1 I & \cdots & \theta(\phi_t(r_q)) \det(\nabla \phi_t(r_q)) \omega_q I \end{array} \right),$$  \hspace{1cm} (6.5)

in which $I$ denotes the $d \times d$ identity matrix.

Remarks. Intuitively, this factorization of $K$ decomposes finite element analysis into natural ingredients: $A$ encodes the combinatorial connectivity of the mesh, $R$ encodes the geometry of the mesh, $S$ encodes the quadrature points, and $D$ encodes the quadrature weights and conductivity. The scaling factor $\alpha_t$, which is necessary for our analysis, cancels out between $D_t$ and $R_t$.

Note that the positive definiteness of $D$ follows from Assumptions 1, 2, and 3. Note also that $A$ is a reduced node-arc incidence matrix of a multigraph defined on the nodes of $T$. (Multigraph indicates that more than one edge may connect a particular pair of vertices.) Each element gives rise to $l - 1$ arcs in the graph. In particular, for each element $t = 1, \ldots, m$, there is an arc joining each of its nodes $2, \ldots, l$ to node 1. Columns corresponding to nodes of $\Gamma_1$ are omitted. (This is what is meant by “reduced.”)

Proof. Let $\xi$ be an arbitrary vector in $\mathbb{R}^n$, and let $u : \hat{\Omega} \rightarrow \mathbb{R}$ be defined by

$$u = \sum_{i=1}^{n} \xi_i \pi_i.$$  \hspace{1cm}

For an element $t$, define function $U_t : T_0 \rightarrow \mathbb{R}$ by

$$U_t = \sum_{\mu=1}^{l} \xi_{LG(t, \mu)} N_{\mu}.$$  \hspace{1cm} (6.6)

With these definitions, it is clear that $u \circ \phi_t = U_t$. In (6.6), we follow the convention that $\xi_i \equiv 0$ in the case that $i \in \{n + 1, \ldots, n'\}$. 11
The assembled stiffness matrix is defined by (3.7) so that

\[ \xi^T K \xi = \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \xi_j K(i,j) \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \xi_j \sum_{t=1}^{m} \sum_{k=1}^{q} \nabla_x \pi_t(\phi_t(r_k)) \cdot \theta(\phi_t(r_k)) \nabla_x \pi_j(\phi_t(r_k)) \det(\nabla \phi_t(r_k)) \omega_k \]

\[ = \sum_{t=1}^{m} \sum_{k=1}^{q} \sum_{i=1}^{n} \xi_i \nabla_x \pi_t(\phi_t(r_k)) \cdot \theta(\phi_t(r_k)) \sum_{j=1}^{n} \xi_j \nabla_x \pi_j(\phi_t(r_k)) \det(\nabla \phi_t(r_k)) \omega_k \]

\[ = \sum_{t=1}^{m} \sum_{k=1}^{q} \nabla_x u(\phi_t(r_k)) \cdot \theta(\phi_t(r_k)) \nabla_x u(\phi_t(r_k)) \det(\nabla \phi_t(r_k)) \omega_k \]

\[ = \sum_{t=1}^{m} \sum_{k=1}^{q} \nabla_x U_t(r_k) \cdot \theta(\phi_t(r_k)) \nabla_x U_1(r_k) \det(\nabla \phi_t(r_k)) \omega_k \]

\[ = \sum_{t=1}^{m} \sum_{k=1}^{q} (\nabla \phi_t(r_k))^{-T} \nabla_x U_t(r_k) \cdot \theta(\phi_t(r_k)) \]

\[ = \nabla^T D \nu. \tag{6.7} \]

In (6.8), we have used the matrix \( D \) defined by (6.4). We have also introduced the vector \( \nu \in \mathbb{R}^{dm} \) defined in block fashion as follows. Write \( \nu = [\nu_1; \cdots; \nu_m] \) where \( \nu_t \in \mathbb{R}^{d} \) is itself composed of blocks \( \nu_t = [\nu_{t,1}; \cdots; \nu_{t,q}] \). Here \( \nu_{t,k} \in \mathbb{R}^{d} \) is defined to be

\[ \nu_{t,k} = (\nabla \phi_t(r_k))^{-T} \nabla_x U_t(r_k)/\alpha_t. \tag{6.9} \]

It is clear by construction of \( D \) and \( \nu \) that \( \nu^T D \nu \) is equal to the expression in (6.7).

Next, we claim that \( \nu = RSA \xi \). Focus on the block corresponding to a particular element \( t \in \{1, \ldots, m\} \) in which case we must show that \( \nu_t = R_t S_{Q,p} A_t \xi \). The product \( A_t \xi \) yields a vector with \( (l-1) \) entries that contains finite differences of entries of \( \xi \). Specifically, the \( \mu - 1 \) entry is \( \xi_{L^G(l,\mu)} - \xi_{L^G(l,1)} \) for \( \mu = 2, \ldots, l \).

By definition of \( S_{Q,p} \) in (4.2), it follows that the \( k \)-block of entries \( (k = 1, \ldots, q) \) of \( S_{Q,p} A_t \xi \) is the \( d \)-vector \( \nabla_x U_t(r_k) \), where

\[ U_t = (\xi_{L^G(l,2)} - \xi_{L^G(l,1)}) N_2 + \cdots + (\xi_{L^G(l,l)} - \xi_{L^G(l,1)}) N_l. \]

Comparing this equation to (6.6) indicates that \( U_t \) and \( U_t \) differ by

\[ \xi_{L^G(l,1)} (N_1 + \cdots + N_l). \]

This latter quantity, however, is a constant function (because \( N_1 + \cdots + N_l \) is identically 1, a property that follows from (3.2) and the fact that 1 is a polynomial of degree at most \( p \) and hence must be expressable as a sum of \( N_p \)’s). Therefore, \( U_t \) and \( U_t \) have the same gradient. We conclude that the \( k \) block of \( S_{Q,p} A_t \xi \) must equal \( \nabla_x U_t(r_k) \).

Combining this equation with (6.3) shows that

\[ R_t S_{Q,p} A_t \xi = (\nabla \phi_t(r_k))^{-T} \nabla_x U_t(r_k)/\alpha_t, \]

and hence is equal to \( \nu_{t,k} \) as defined by (6.9). This concludes the proof that \( \nu = RSA \xi \).
Let us define \( J_\alpha \). For the last line, we used the fact that.

The last line follows from (4).

Now we analyze the singular values of \( J \) to finish the first part of Theorem 5.1. Let us define \( J = RS \) so that \( K = A^T J^T D J A \) as claimed. The block structures of \( R \) and \( S \) induce a corresponding block structure on \( J \):

\[
J = \begin{pmatrix} J_1 & \cdots & J_m \end{pmatrix}
\]

where

\[
J_t = R_t S_{Q,p}
\]

for all \( t = 1, \ldots, m \). Because of this structure, the maximum singular value of \( J \) is the maximum singular value among any of its blocks and similarly for its minimum singular value. Since in general \( \sigma_{\max}(AB) \leq \sigma_{\max}(A)\sigma_{\max}(B) \),

\[
\sigma_{\max}(J_t) \leq \sigma_{\max}(R_t)\sigma_{\max}(S_{Q,p})
\]

\[
= \sigma_{\max}(\text{diag}(\nabla \phi_t(r_1)^{-T}, \ldots, \nabla \phi_t(r_q)^{-T}))\sigma_{Q,p}/\alpha_t
\]

\[
= \max_{k=1, \ldots, q} \sigma_{\max}(\nabla \phi_t(r_k)^{-1})\sigma_{Q,p}/\alpha_t
\]

\[
= \sigma_{Q,p}.
\]

The last line follows from (4.4) and establishes (5.1).

Since \( \sigma_{\min}(AB) \geq \sigma_{\min}(A)\sigma_{\min}(B) \) for two matrices \( A, B \) with full column rank,

\[
\sigma_{\min}(J_t) \geq \sigma_{\min}(R_t)\sigma_{\min}(S_{Q,p})
\]

\[
= \sigma_{\min}(\text{diag}(\nabla \phi_t(r_1)^{-T}, \ldots, \nabla \phi_t(r_q)^{-T}))\tau_{Q,p}/\alpha_t
\]

\[
= \min_{k=1, \ldots, q} \sigma_{\min}(\nabla \phi_t(r_k)^{-1})\tau_{Q,p}/\alpha_t
\]

\[
= \min_{k=1, \ldots, q} (1/\sigma_{\max}(\nabla \phi_t(r_k)))\tau_{Q,p}/\alpha_t
\]

\[
= (1/\beta_t) \cdot \tau_{Q,p}/\alpha_t
\]

\[
\geq \tau_{Q,p}/\kappa_1(T).
\]

For the last line, we used the fact that \( \alpha_t\beta_t \leq \kappa_1(T) \), which follows from (4.6). This establishes (5.2) and concludes the proof of the first factorization in Theorem 5.1.

Our factorization \( K = A^T J^T D J A \) is reminiscent of one proposed by Argyris [1] of the form \( K = APA^T \), which he calls the “natural factorization.” In Argyris’s factorization, however, the matrix \( \bar{A} \) has all +1 and 0 entries and therefore is not a node-arc incidence matrix. The purpose of Argyris’s matrix \( \bar{A} \) is to assemble the element stiffness matrices, which constitute the blocks of the block-diagonal matrix \( P \).

7. The second factorization. In this section we prove the second part of Theorem 5.1. We state the existence of the second factorization in the form of a lemma.

Lemma 7.1. Let \( J \) and \( D \) be defined as in Lemma 6.1. Then \( J^T D J \) can be refactored as \( D^{1/2} J^T J D^{1/2} \), where
Matrix $\bar{D} \in \mathbb{R}^{m(l-1) \times m(l-1)}$ is positive definite diagonal and is written in block form

$$\bar{D} = m_Q \begin{pmatrix} f_1 g_1 \alpha_t^2 I & \cdots & \cdots & f_m g_m \alpha_t^2 I \\ & & & \\ & & & \\ \end{pmatrix}$$

(7.1)

where $I$ is the $(l-1) \times (l-1)$ identity matrix. In this formula, $f_t$ is the minimum value of $\theta$ over Gauss points of element $t$:

$$f_t = \min_{k=1,\ldots,q} \theta(\phi_t(r_k)).$$

(7.2)

and similarly, $g_t$ is the minimum value of $\det(\nabla \phi_t)$:

$$g_t = \min_{k=1,\ldots,q} \det(\nabla \phi_t(r_k)).$$

(7.3)

Matrix $\bar{J} \in \mathbb{R}^{dqm \times (l-1)m}$ is defined by

$$\bar{J} = D^{1/2} \bar{D}^{-1/2}.$$  

(7.4)

**Proof.** The fact that $\bar{D}^{1/2} J^T J \bar{D}^{1/2} = J^T D J$ follows as an immediate consequence of (7.4) regardless of how we have defined $\bar{D}$. \(\Box\)

Next we analyze the condition number of $H$ to finish proving the second part of Theorem 5.1. We define $H = J^T \bar{J}$ so that $J^T D J = \bar{D}^{1/2} H \bar{D}^{1/2}$. We must estimate the singular values of $\bar{J}$, which are the square roots of the eigenvalues of $H$.

Let the diagonal block of $\bar{J}$ associated with element $t$ be denoted $\bar{J}_t$ for $t = 1, \ldots, m$. Because of the block structure, the maximum and minimum singular values for $\bar{J}$ are the maximum and minimum singular values among the blocks $\bar{J}_t$ which may be written

$$\bar{J}_t = D_t^{1/2} J_t f_t^{-1/2} g_t^{-1/2} m_Q^{-1/2} \alpha_t^{-1} = \bar{D}_t J_t,$$

where

$$\bar{D}_t = f_t^{-1/2} g_t^{-1/2} m_Q^{-1/2} \alpha_t^{-1} D_t^{1/2}.$$

Examining the constituent parts of $D_t$ given by (6.5), observing that the $\alpha_t^2$ in (6.5) is cancelled by the $\alpha_t^{-1}$ in the preceding equation and applying the inequalities

$$1 \leq \omega_k / m_Q \leq M_Q / m_Q$$

for $k = 1, \ldots, q$ (see (4.1)),

$$1 \leq \theta(\phi_t(r_k)) f_t^{-1} \leq \frac{\max_{k=1,\ldots,q} \theta(\phi_t(r_k))}{\min_{k=1,\ldots,q} \theta(\phi_t(r_k))}$$

(see (7.2)), and

$$1 \leq \det(\nabla \phi_t(r_k)) g_t^{-1} \leq \frac{\max_{k=1,\ldots,q} \det(\nabla \phi_t(r_k))}{\min_{k=1,\ldots,q} \det(\nabla \phi_t(r_k))}$$
(see (7.3)), we conclude that the diagonal entries of $\hat{D}_t$ satisfy

$$1 \leq \hat{D}_t(i, i) \leq \left( \max_k \theta(\phi_t(r_k)) \right)^{1/2} \left( \min_k \det(\nabla \phi_t(r_k)) \right)^{1/2} \left( \frac{M_Q}{m_Q} \right)^{1/2}.$$ 

By (4.7) and (4.8), the first two quantities on the right-hand side of the preceding equation are bounded by $\hat{\theta}(T)^{1/2}$ and $\kappa_2(T)^{1/2}$ respectively. Thus, it is apparent that for each $i$,

$$1 \leq D_t(i, i) \leq (\hat{\theta}(T) \kappa_2(T) M_Q/m_Q)^{1/2}.$$ 

Since $J_t = \bar{D}_t J_t$, we can combine the inequalities in the previous line with (6.11) and (6.12) to obtain:

$$\sigma_{\text{max}}(\bar{J}_t) \leq \hat{\theta}(T)^{1/2} \kappa_2(T)^{1/2} M_Q^{1/2} \sigma_{\varphi,p} m_Q^{1/2},$$  

and

$$\sigma_{\text{min}}(\bar{J}_t) \geq \tau_{\varphi,p}/\kappa_1(T).$$  

Since $H = J^T J$ and $J$ has full column rank, (7.5) and (7.6) prove (5.3), which concludes the proof of Theorem 5.1.

8. Preconditioning Strategy Summary. The main result of this paper is that the stiffness matrix $K$ of (3.1) can be approximated by a symmetric diagonally dominant matrix $\bar{K}$. In this section we discuss several approaches for using this approximation to efficiently solve the linear system $Kx = f$ by iteration.

The first step in using the approximation to construct $\bar{K}$. Recall that $\bar{K} = A^T \bar{D} A$, and thus $A$ and $\bar{D}$ must be formed. Forming $A$ means construction of the multigraph consisting of $l-1$ edges (a star-tree) per element of $T$. Matrix $A$ is then the reduced node-arc incidence matrix of this multigraph as specified by Lemma 6.1. Construction of $\bar{D}$ is given by (7.1). Notice that computing $A$ and $\bar{D}$ requires information about the mesh and original boundary value problem. In other words, our method is not applicable (at least not in a straightforward manner) if the only information about the original problem is the stiffness matrix $K$.

Once $\bar{K}$ is on hand, there are several ways to proceed. The most straightforward is to suppose that one has an efficient preconditioned conjugate gradient solver for systems of the form $Kx = f$. Efficient preconditioners for symmetric diagonally dominant systems were proposed and analyzed in a graph-theoretic framework by Vaidya. Vaidya’s work is described and extended by [4, 6], and [11] contains a related analysis. One of the most recent improvements is due to Spielman and Teng [20], who propose a graph-based preconditioner whose running time is $O(n^{5/4})$. (Spielman and Teng improve this bound to $O(n^{1+\epsilon})$ for any $\epsilon > 0$, but the algorithm corresponding to the improved bound is no longer preconditioned conjugate gradients.) Other techniques proposed in the literature for symmetric diagonally dominant matrices such as algebraic multigrid could also be used.

If one has a good preconditioner $M$ for $\bar{K}$, then one could also use $M$ as a preconditioner directly for $K$. This is because of the “triangle inequality” (see [5]), which states

$$\sigma(K, M) \leq \sigma(K, \bar{K}) \sigma(\bar{K}, M).$$  

(8.1)
This shows that the the overall support number is bounded by the product of the support numbers in each step of the approximation $K \approx \tilde{K} \approx M$. In particular, $\kappa(M, K) \leq \kappa(H)\kappa(M, \tilde{K})$. Since the number of iterations of preconditioned conjugate gradients is bounded by the square root of the condition number, our analysis shows that the increase in the number of iterations for solving $K\hat{x} = f$ (compared to solving $\tilde{K}\hat{x} = f$) is at most a factor of $\sqrt{\kappa(H)}$, which in turn is bounded by

$$\hat{O}(T)^{1/2}\kappa_1(T)\kappa_2(T)^{1/2}M\sigma Q_\rho / (m Q_\rho p).$$

The asymptotically fastest known iterative algorithm for solving symmetric diagonally dominant matrices is due to Elkin et al. [9]. This algorithm, which we denote EEST, extends Spielman and Teng [20] and requires $O(n_s(\log n)^s)$ time to solve any diagonally dominant $K\hat{x} = f$, where $n_s$ is the number of nonzero entries in $K$ and $s$ is some constant. In our case, $n_s = O(n)$. The EEST iteration is more complicated than preconditioned conjugate gradients, and hence it is unclear whether it is possible to simply replace the diagonally dominant coefficient matrix $K$ by our stiffness matrix $\tilde{K}$ and expect the algorithm to converge in $O(n(\log n)^s\kappa(H)^{1/2})$ time.

On the other hand, one could obtain this running time $O(n(\log n)^s\kappa(H)^{1/2})$ by using a nested iteration: the outer iteration is preconditioned conjugate gradients in which $\tilde{K}$ preconditions $K$. Thus, $O(\kappa(H)^{1/2})$ outer iterations are required. The inner loop is the EEST algorithm to apply the preconditioner, i.e., to solve $\tilde{K}\hat{x} = f$ in $O(n(\log n)^s)$ iterations. The EEST algorithm is fairly complex, and using it in a two-level manner would raise a number of difficulties associated with termination tests, so it is unlikely to be practical currently.

This bound of $O(n(\log n)^s\kappa(H)^{1/2})$ on the number of operations is independent of the condition number of the underlying system because, as noted above $\kappa(H)$ depends on factors associated with the finite element method and on mesh quality measures but not on the conditioning of $K$.

Our approximation scheme can be rewritten on an element-by-element basis as follows. For some element index $t \in \{1, \ldots, m\}$, consider its element stiffness matrix $K_t = A_t^T J_t^2 D_t J_t A_t$, where $A_t$ is defined in Lemma 6.1, $J_t$ is defined by (6.10), and $D_t$ is defined by (6.5). The above proof shows that $\tilde{D}_t$, which is the $t$th block of (7.1), is a good approximation to $J_t^2 D_t J_t$. Thus, we let $\tilde{K}_t = A_t^T \tilde{D}_t A_t$. The overall approximation is $\tilde{K} = \sum_{t=1}^m \tilde{K}_t$. Note that the so-called splitting lemma frequently used in support-graph theory (see, e.g., [5]) shows that

$$\sigma(K, \tilde{K}) \leq \max(\sigma(K_1, \tilde{K}_1), \ldots, \sigma(K_m, \tilde{K}_m)),$$

so it suffices to analyze the quality of approximation of $\tilde{K}_t$ to $K_t$.

Alternatively, we can take a global view as in the above analysis and write $\tilde{K} = A^T \tilde{D} A$, where $A = (A_1; A_2; \ldots; A_m)$ and $\tilde{D} = \text{diag}(\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_m)$. To simplify notation, we have adopted the global view in this paper, but the reader should keep in mind that our approximation can take place element by element, which may be important in an implementation.

9. **Notes added in revision.** Since the first version of this paper, there have been three related developments of interest. First, R. Gupta’s Master’s thesis [12] gave a geometric method to approximate element stiffness matrices by diagonally dominant matrices in the $p = 1$, $d = 2$ case. Gupta’s construction bears some similarity to ours, although a bound like (5.3) is not obtained. Computational experiments are performed.
There are at least three rele-

These cases were tried, which we denote \( T_1, T_2, T_3, T_4, T_5, T_6 \), i.e., linear and quadratic triangles and linear and quadratic tetrahedra. Six test cases are considered: \((1/3, 1/3, 1/2)\), \((1/6, 1/6, 1/6)\), \((1/6, 2/3, 1/6)\), \((2/3, 1/6, 1/6)\)

Table 10.2

<table>
<thead>
<tr>
<th>Test</th>
<th>( d )</th>
<th>( p )</th>
<th>( q )</th>
<th>( \sigma_{Q,p} )</th>
<th>( \tau_{Q,p} )</th>
<th>( M_{Q}/m_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_1 )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>( T_2 )</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5.26</td>
<td>0.83</td>
<td>1.00</td>
</tr>
<tr>
<td>( T_3, T_5 )</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>( T_4, T_6 )</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>6.47</td>
<td>0.63</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 10.1

Quadrature rules used for the six test cases. The rules are written in the form \((r_1, \omega_1), \ldots, (r_q, \omega_q)\), where \( r_k \) is a Gauss point in the reference simplex and \( \omega_k \) is its weight. In this table \( \xi_1 = (10 - \sqrt{20})/40 \) and \( \xi_2 = 1 - 3\xi_1 \).

Second, Phillips and Miller [18] have shown that in the \( p = 1, d = 2 \) case, \( \kappa(H) \) is still small even if some elements have very small angles, provided that no element has large angles (close to \( \pi \)). In our preceding analysis, the right-hand side of (5.3) would grow large in the presence of small angles since the factor \( \kappa_1(T) \) would be large. Their result probably can be extended to the \( p \geq 2 \) case.

In a third development, Avron et al. [2] have shown how to get an approximation to element stiffness matrices that is optimal up to a constant factor. In other words, for an element stiffness matrix \( K_t \) they find a diagonally dominant matrix \( \tilde{K}_t \) such \( \kappa(K_t, \tilde{K}_t) \leq c_1 \kappa(K_t, K'_t) \), where \( K'_t \) is any other symmetric diagonally dominant matrix of the correct size and \( c_1 \) depends only on \( p, d \). Thus, their preconditioner could lead to a faster algorithm than ours since ours is not optimal in this sense. Their theory, however, does not subsume ours since they do not obtain any new bounds on \( \kappa(K, \tilde{K}) \) comparable to (5.3).

10. Computational tests. There are at least two possible ways to test the effectiveness of our result: calculate the quality of matrix approximation or measure the speed of an iterative method. Since our theory is primarily about the former issue, and since Avron et al. conduct extensive testing on the latter question, we focus on the first question.

We try out four specific commonly occurring cases: \((d, p) = (2, 1), (2, 2), (3, 1), (3, 2)\) i.e., linear and quadratic triangles and linear and quadratic tetrahedra. Six test meshes were tried, which we denote \( T_1, \ldots, T_6 \); the first two are two-dimensional and the last four are three-dimensional. For quadrature in the \( p = 1 \) cases, we use the midpoint rule. For the \( p = 2 \) cases, we use the symmetric \( d + 1 \)-point rule, which is accurate for polynomials up to degree 2. Table 10.1 gives the quadrature rules used.

Table 10.2 gives the values of relevant scalars for these test cases. It should be noted that \( \sigma_{Q,p} = \tau_{Q,p} = 1 \) for the \( p = 1 \) cases under the midpoint rule because in these cases \( S_{Q,p} \) given by (4.2) turns out to be the identity matrix.

Finally, we can tabulate element stiffness approximation bounds. (As mentioned in Section 8, it suffices to measure element stiffness approximation, which is much easier to compute than global stiffness approximation.) There are at least three rele-
Table 10.3
Approximation of the element stiffness matrix by a diagonally dominant matrix.

<table>
<thead>
<tr>
<th>Test</th>
<th>m</th>
<th>n</th>
<th>$\kappa_1(T)$</th>
<th>$\kappa_2(T)$</th>
<th>$\chi_1$</th>
<th>$\chi_2$</th>
<th>$\chi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>234</td>
<td>143</td>
<td>4.1</td>
<td>1.0</td>
<td>16.6</td>
<td>16.6</td>
<td>16.6</td>
</tr>
<tr>
<td>$T_2$</td>
<td>234</td>
<td>520</td>
<td>4.7</td>
<td>1.3</td>
<td>185.1</td>
<td>185.1</td>
<td>1117.2</td>
</tr>
<tr>
<td>$T_3$</td>
<td>9078</td>
<td>1913</td>
<td>59.6</td>
<td>1.0</td>
<td>3549.5</td>
<td>3549.5</td>
<td>3549.5</td>
</tr>
<tr>
<td>$T_4$</td>
<td>9078</td>
<td>13608</td>
<td>2.7</td>
<td>$5.96 \cdot 10^4$</td>
<td>3549.5</td>
<td>3549.5</td>
<td>3549.5</td>
</tr>
<tr>
<td>$T_5$</td>
<td>729</td>
<td>220</td>
<td>5.0</td>
<td>1.0</td>
<td>24.9</td>
<td>24.9</td>
<td>24.9</td>
</tr>
<tr>
<td>$T_6$</td>
<td>729</td>
<td>1330</td>
<td>5.0</td>
<td>1.2</td>
<td>396.6</td>
<td>396.6</td>
<td>2644.5</td>
</tr>
</tbody>
</table>

Vant quantities to tabulate: $\chi_1 = \kappa(K_t, \bar{K}_t)$, $\chi_2 = \kappa(H)$, and $\chi_3 = \theta(T)\kappa_1(T)^2\kappa_2(T)\cdot M_{\omega\sigma^2_{\omega,\infty}}$, which is the right-hand side of (5.3). Here, $K_t$ is the stiffness matrix of element $t$ ($t = 1, \ldots, m$), and $\bar{K}_t$ is the diagonally dominant preconditioner given by $A^T_t\bar{D}_tA_t$. It follows from Theorem 2.3 and (5.3) that $\chi_1 \leq \chi_2 \leq \chi_3$.

We generate six meshes and take the max of $\chi_1, \chi_2, \chi_3$ over all elements of the mesh. For conductivity, we use $\theta \equiv 1$, so that the factor $\theta(T)$ does not enter the bound. The meshes are generated as follows. In two dimensions, we use the Triangle mesh generator [19] to generate a mesh of an annulus, which is $T_1$ and consists of linear ($p = 1$) elements. To generate $T_2$, a quadratic mesh, we insert midpoint nodes in every element of $T_1$, and for the edges on the boundaries, we moved the midpoint nodes onto the boundary.

The remaining four meshes are three-dimensional. For $T_3$, we used the QMG mesh generator [17] to generate a mesh of a unit ball in $\mathbb{R}^3$. QMG’s meshes are intended for the $p = 1$ case. For $T_4$ we use midpoint insertion on the $T_3$ mesh to obtain a $p = 2$ mesh. Again, midpoint nodes of edges whose endpoints were on the boundary were displaced outward onto the boundary. For $T_5$ and $T_6$, we again mesh a unit ball (actually, one octant of the ball) using a mesh generator tailored for that domain only. The mesh generator for $T_5$ first generates a highly regular mesh of a unit tetrahedron, and then it projects the mesh nodes radially outward toward one facet so that an octant of the unit ball is covered. Finally, $T_6$ is obtained from $T_5$ using midpoint insertion and displacement at the boundary.

Then for every element in each mesh, we compute the three quantities $\chi_1, \chi_2, \chi_3$. We have tabulated the maximum values in Table 10.3. The undesirably large values of $\chi_1$ for $T_3$ and $T_4$ appears to be due primarily to poorly shaped elements, i.e., large value of $\kappa_1(T)$. This is evident from comparing $T_3$ and $T_4$ against $T_5$ and $T_6$, two meshes for the same domain but with much better shaped tetrahedra.

The value of $\kappa_2(T)$ was quite small in all tests; as mentioned earlier, it is identically 1 for the $p = 1$ cases. For the $p = 2$ cases, interior elements are still linear, and elements adjacent to the boundary are fairly close to linear. It is interesting to note that $\chi_1 = \chi_2$ in all cases. This implies that the invariant subspaces of $H$ corresponding to its extremal eigenvalues meet the range space of $\bar{D}^{1/2}A$. We do not have a deeper explanation for this observation.

Although an iterative method was not tested, some conclusions can still be drawn from Table 10.3. For example, if our approximation were used on $T_6$ in an iterative setting, then the slowdown would be at most a factor of $\sqrt{\kappa(K_t, K_t)}$, i.e., a slowdown of at most a factor of 20. In practice, the method seems to work better than that according to Avron et al.
11. **Open questions.** This work is the first to extend support-tree methods, which previously have been shown to be good preconditioners for diagonally dominant matrices with negative off-diagonal entries, to the class of finite element matrices. We have shown that the scope of the method includes a standard scalar elliptic boundary value problem, but perhaps the scope of finite element problems that can be tackled with this method could be expanded further.

One generalization would be the class of problems $\nabla \cdot (\Theta(x) \nabla u) = -f$, where $\Theta(x)$ is a spatially varying $d \times d$ symmetric positive definite matrix. This generalization would present problems for our current analysis in the case that $\Theta(x)$ is highly ill-conditioned. It would still be straightforward to write $K = A^T J^T D J A$ where $D$ is now block diagonal, but our analysis of the introduction of $\bar{J}$ would run into trouble because the $dq \times dq$ diagonal blocks of $D$ are no longer individually well conditioned.

It would also be interesting to tackle vector problems such as linear elasticity or Stokes’ flow, or higher-order equations like the biharmonic equation. It seems likely that our techniques can extend to at least some of these problems since they all have a symmetric positive definite weak form. On the other hand, a recent result [7] suggests that the extension of our results to linear elasticity will not be straightforward because of the higher nullity of element stiffness matrices in the case of linear elasticity. A further generalization would be to unsymmetric problems like the convection-diffusion equation. The latter class of problems would require substantial rethinking of the whole approach since condition number reduction, which is very relevant for the application of conjugate gradients to symmetric positive definite systems, is less relevant to the application of GMRES to unsymmetric systems.

Our analysis is based on condition numbers (support numbers). One drawback of this approach is that the convergence and work estimates may be too pessimistic. For instance, the condition number of the preconditioned linear systems depends on $\kappa_1(T)$, the worst aspect ratio of any element in the mesh. If there is only one poorly shaped element in the mesh, we expect iterative solvers will only take a few extra iterations since changing a single element implies a low-rank correction to the assembled stiffness matrix. Any analysis based on condition numbers will be unable to capture this effect. A related open issue is whether we can exploit recent work in mesh quality metrics [16] to show that “good meshes” both have small error in the FEM approximation and also produce linear systems that can be well approximated by diagonally dominant systems.

Another point to make about our method is that, although the condition number of the preconditioned system has an upper bound independent of

$$R_\theta = \max_{x \in \Omega} \theta(x)/\min_{x \in \Omega} \theta(x),$$

there will still be a loss of significant digits due to roundoff error when using our method in the case that $R_\theta$ is large. This is because the system matrix and the preconditioner separately are ill-conditioned operators. A special case of (3.1) in which $R_\theta$ is extremely large and in which $f = 0$ was considered in [22]. That paper proposed a method based on Gaussian elimination in which the loss of significant digits is avoided. Some of the ideas behind [22] were also extended to solution via conjugate gradient using support preconditioners by [14]. The methodology in [14], however, was for node-arc incidence matrices and for a particular kind of support preconditioner called a support tree [11], and it is not clear whether that method would apply to the present setting.
Acknowledgments. The authors wish to thank the three anonymous referees of this paper for their very helpful comments.

REFERENCES