SAND REPORT

SAND2004-3310
Unlimited Release
Printed July 2004

An Analytically Solvable Eigenvalue Problem for the Linear Elasticity Equations

David Day, Louis Romero

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia is a multiprogram laboratory operated by Sandia Corporation,
a Lockheed Martin Company, for the United States Department of Energy's
National Nuclear Security Administration under Contract DE-AC04-94-AL85000.

Sandia National Laboratories

Issued by Sandia National Laboratories, operated for the United States Department of Energy by Sandia Corporation.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from
U.S. Department of Energy
Office of Scientific and Technical Information
P.O. Box 62
Oak Ridge, TN 37831

Telephone:	(865) 576-8401
Facsimile:	(865) 576-5728
E-Mail:	reports@adonis.osti.gov
Online ordering:	http://www.doe.gov/bridge

Available to the public from
U.S. Department of Commerce
National Technical Information Service
5285 Port Royal Rd
Springfield, VA 22161

Telephone:	(800) 553-6847
Facsimile:	(703) 605-6900
E-Mail:	orders@ntis.fedworld.gov
Online ordering:	http://www.ntis.gov/help/ordermethods.asp?loc=7-4-0#online

SAND2004-3310
Unlimited Release
Printed July 2004

An Analytically Solvable Eigenvalue Problem for the Linear Elasticity Equations

David Day, Louis Romero
Computational Mathematics and Algorithms Department
Sandia National Laboratories
P.O. Box 5800
Albuquerque, NM 87185-1110

July 19, 2004

Abstract

Analytic solutions are useful for code verification. Structural vibration codes approximate solutions to the eigenvalue problem for the linear elasticity equations (Navier's equations). Unfortunately the verification method of "manufactured solutions" does not apply to vibration problems. Verification books (for example [2]) tabulate a few of the lowest modes, but are not useful for computations of large numbers of modes. A closed form solution is presented here for all the eigenvalues and eigenfunctions for a cuboid solid with isotropic material properties. The boundary conditions correspond physically to a greased wall.

Acknowledgments

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

An Analytically Solvable Eigenvalue Problem for the Linear Elasticity Equations

Contents

1 Introduction

Structural vibration codes approximate solutions to the eigenvalue problem for the linear elasticity equations (Navier's equations). Here the boundary conditions would apply for example at a greased wall. Homogeneous boundary conditions apply to the normal component of displacement and the tangential components of stress vector ($\mathbf{t}^{\mathbf{T}}\sigma\mathbf{n}$). The modal density for specific boundary conditions is compared to the asymptotic spectral density.

2 The Equations of Linear Elasticity

The notation is based on [6], but is more contemporary as in [3]. The most common mistake in using the analytic solutions comes from translating between different notations.

All functions are assumed to be smooth. The displacement vector $\mathbf{u}$ has ith component u_i. The linearized strain tensor ϵ is a (3 by 3 symmetric) matrix valued function with components

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In summation notation ϵ_{kk} is the trace of the strain tensor. The linearized stress tensor σ is another (3 by 3 symmetric) matrix valued function with components

$$\sigma_{ij} = 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}\right) + K\delta_{ij}\epsilon_{kk}. = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}.$$

$\lambda = K - \frac{2\mu}{3}$ and μ are called Lamé's constants, and K is called the modulus of compression. We assume that μ and $\lambda + \mu$ are positive. The equilibrium equation for the strain and external body forces $\mathbf{f}\ dV$ is Hooke's Law, div$\sigma + \mathbf{f} = \mathbf{0}$. Newton's Law states that $\rho\frac{\partial^2\mathbf{u}}{\partial t^2} + \mathbf{f} = \mathbf{0}$.

The stress tensor is most often expressed in terms of Young's modulus E and Poisson's ratio ν. With respect to Lamé's constants and K,

$$E = \frac{9K\mu}{3K+\mu} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \text{and } \nu = \frac{1}{2}\frac{3K-2\mu}{3K+\mu} = \frac{\lambda}{2(\lambda+\mu)}.$$

Conversely

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \text{and } \mu = \frac{E}{2(1+\nu)}.$$

The relationships between the different constants are tabulated in [5], Table 4.1 page 47.

Some prefer to write the PDE in divergence form, $\mathrm{div}\ (A. * D\mathbf{u} + B. * (FD\mathbf{u}F)) = \mathbf{u}\xi$ for flip matrix $F = [0, 1; 1, 0]$ and coefficient matrices $A = [2\mu + \lambda, \mu; \mu, 2\mu + \lambda]$ and $B = [\lambda, \mu; \mu, \lambda]$.

Elastic waves are described by the linear system

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\mu}{\rho}\Delta \mathbf{u} + \frac{\lambda+\mu}{\rho}\mathrm{grad}\ \mathrm{div}\ \mathbf{u} =$$

$$= \frac{\lambda+\mu}{\rho}\left(\frac{\mu}{\lambda+\mu}\nabla^2\mathbf{u} + \nabla\nabla\cdot\mathbf{u}\right) =$$

$$\frac{E}{2\rho(1+\nu)(1-2\nu)}\left((1-2\nu)\nabla^2\mathbf{u} + \nabla\nabla\cdot\mathbf{u}\right).$$

Monochromatic acoustic waves have (u, v, w) components

$$\phi(\mathbf{x}, t) = e^{i\omega t + i\mathbf{k}\cdot\mathbf{x}}, \quad \frac{\partial^2\phi}{\partial t^2} = c^2\Delta\phi.$$

The Hodge decomposition $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s$ such that $\mathrm{curl}\ \mathbf{u}_p = \mathbf{0}$, $\mathrm{div}\ \mathbf{u}_s = 0$ decomposes elastic waves into pressure and shear waves

$$\frac{\partial^2\mathbf{u}_p}{\partial t^2} = c_p^2\Delta\phi, \quad \frac{\partial^2\mathbf{u}_s}{\partial t^2} = c_s^2\Delta\phi.$$

The pressure and shear wave speeds are

$$c_s^2 = \frac{\mu}{\rho}, \quad c_p^2 - c_s^2 = \frac{\lambda+\mu}{\rho}. \tag{2.1}$$

Monochromatic pressure and shear waves are solutions

$$\mathbf{u}_p = \mathbf{a}_p e^{i\omega t + i\mathbf{k}_p\cdot\mathbf{x}}, \quad \mathbf{u}_s = \mathbf{a}_s e^{i\omega t + i\mathbf{k}_s\cdot\mathbf{x}},$$

$$\mathbf{a}_p = \gamma_p\mathbf{k}_p, \omega = c_p k_p \quad \text{or} \quad \mathbf{a}_s\cdot\mathbf{k}_s = 0, \omega = c_s k_s \tag{2.2}$$

2.1 Spectral Density

An elliptic partial differential equation with eigenvalues $(\lambda_n)_{n>0}$ and frequencies $\omega_n = \sqrt{\lambda_n}$ has spectral density

$$N(\omega) = \sum_{\omega_n<\omega} 1 + \sum_{\omega_n=\omega}\frac{1}{2}.$$

Zero modes are not counted. The $1/2$ is there so that at discontinuities the value of $N(\omega)$ is the average of the left and right limits.

The asymptotic spectral density does not depend on the boundary conditions, and for the elastic wave equation ([1]),

$$\ddot{u} = c_s^2\, \Delta \mathbf{u} + (c_p^2 - c_s^2)\, \mathrm{grad}\ \mathrm{div}\ \mathbf{u}$$

over a domain with volume V

$$N(\omega) = \left(2c_s^{-3} + c_p^{-3}\right) \frac{V\omega^3}{6\pi^2} + \mathcal{O}(\omega^{5/2}(\log \omega)^{1/2}).$$

2.2 Easy Case: Plain Strain

We consider a two dimensional tensor product domain (plain strain linear elasticity, $\mathbf{u}^T = (u,v)(x,y)$) over the domain $\Omega = [0,l_1] \times [0,l_2]$.

A general wave number $\mathbf{k}$ corresponds to eigenfunctions $\mathbf{u} = \mathbf{a}e^{i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ of

$$c_s^2 \Delta \mathbf{u} + (c_p^2 - c_s^2)\mathrm{grad}\ \mathrm{div}\ \mathbf{u} = -\mathbf{u}\xi. \tag{2.3}$$

The secular equation is $\mathbf{a}c_s^2 + \mathbf{k}(\mathbf{k}^T\mathbf{a})c_p^2 = \mathbf{a}\xi$. Such eigenfunctions are either pressure or shear waves of the form described in equation (2.2).

For the domain $\Omega = [0,\pi]^2$, a general pressure eigenfunction has the form

$$\mathbf{u} = \sum_{1 \le j \le 4} \gamma_j k_j e^{i\mathbf{k}_j\cdot\mathbf{x}},$$

where

$$[k_1, k_2, k_3, k_4] = \begin{bmatrix} m & m & -m & -m \\ n & -n & n & -n \end{bmatrix}$$

Completeness of the eigenfunctions of equation (2.3) is addressed in the last section. What boundary conditions are reachable? A nontrivial equation of this form does not satisfy $\mathbf{u}(0,y) = \mathbf{0}$. We will show next that it can satisfy either $\mathbf{u}\cdot\mathbf{n} = 0$ or $\mathbf{u}\cdot\mathbf{t} = \mathbf{0}$.

Take the case $\mathbf{u}\cdot\mathbf{n} = 0$. We want $u(0,y) = 0$ and $v(x,0) = 0$. Some elementary algebra leads to solutions

$$\mathbf{u}^T(x,y) = (m \sin mx \cos ny, n \cos mx \sin ny).$$

The shear waves are similar. The eigenfunctions are complete for the greased wall boundary conditions (see last section). We shall see that in the $2D$ case for example with $1 = \rho = \lambda = \mu = L_x = L_y$, the pressure eigenvalues are $3(n^2+m^2)\pi^2$ for $n,m \ge 1$ and the shear eigenvalues are $(n^2+m^2)\pi^2$ for $n,m >= 0$ and $(n,m) \ne \mathbf{0}$.

The boundary condition $\mathbf{u}\cdot\mathbf{t} = \mathbf{0}$ implies that that $u(x,0) = 0$ and $v(0,y) = 0$.

$$\mathbf{u}^T(x,y) = (m \cos(mx) \sin(ny), n \sin(mx) \cos(ny)).$$

Other boundary conditions are satisfied by “scrambled” waves ([1]). An example with stress free boundary conditions is treated in in [4].

3 An Eigenvalue Problem

In the following, $\mathbf{x}^T = (x, y, z)$ and $\mathbf{u}^T = (u, v, w)$. We choose the hexahedral domain $\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$. We also choose the boundary conditions:

$$u = \sigma_{xy} = \sigma_{xz} = 0 \quad \text{if} \quad x \in \{0, L_x\},$$

$$v = \sigma_{xy} = \sigma_{yz} = 0 \quad \text{if} \quad y \in \{0, L_y\},$$

$$w = \sigma_{xz} = \sigma_{yz} = 0 \quad \text{if} \quad z \in \{0, L_z\}.$$

The subscripts denote components. Equivalently,

$$\text{(3.1)} \qquad \begin{cases} u = \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0 & \text{if} \quad x \in \{0, L_x\}, \\ v = \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} = 0 & \text{if} \quad y \in \{0, L_y\}, \\ w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 & \text{if} \quad z \in \{0, L_z\}. \end{cases}$$

If we assume a time dependence of the form $e^{i\omega t}$ the function $\mathbf{u}$ must satisfy the eigenvalue problem

$$\text{(3.2)} \qquad (1 - 2\nu)\nabla^2\mathbf{u} + \nabla\nabla \cdot \mathbf{u} = -\mathbf{u}\xi.$$

along with the boundary conditions in (3.1). The frequencies ω are determined from

$$\rho\omega^2 = \xi \frac{E}{2(1 + \nu)(1 - 2\nu)}$$

4 An Ansatz for the Eigenfunctions

Before discussing eigenfunctions, we figure out a basis for the functions satisfying the boundary conditions (3.1). Then we will find the eigenfunctions with simple representations in the basis. The proof of completeness is post-poned until the next section.

The functions $(\sin(nx))_{n>0}$ are a basis for the smooth univariate functions vanishing at 0 and π. And the univariate functions $(\cos(nx))_{n\geq 0}$ are a basis for the smooth functions whose derivative vanishes at 0 and π. The bases for components of an arbitrary function $\mathbf{u}$ satisfying (3.1) are similar. For each coordinate axis, we choose either $\sin()$ or $\cos()$, making for a total of eight possibilities. Each interval is transformed to $[0, \pi]$, using

$$\alpha_l = \frac{l\pi}{L_x}, \quad \beta_m = \frac{m\pi}{L_y}, \quad \gamma_n = \frac{n\pi}{L_z}, \quad l, m, n = 0, 1, 2, \ldots.$$

The bases for u, v and w are respectively (f_{lmn}), (g_{lmn}) and (h_{lmn}) where

$$f_{lmn}(\mathbf{x}) = \sin(\alpha_l x)\cos(\beta_m y)\cos(\gamma_n z),$$

$$g_{lmn}(\mathbf{x}) = \cos(\alpha_l x)\sin(\beta_m y)\cos(\gamma_n z),$$
$$h_{lmn}(\mathbf{x}) = \cos(\alpha_l x)\cos(\beta_m y)\sin(\gamma_n z).$$

We seek eigenfunctions with same value of (l, m, n) for each component,

$$\mathbf{f}_{lmn} = (f_{lmn}, g_{lmn}, h_{lmn}).$$

The symbol $.*$, denoting the component wise or Hadamard product of two vectors, is very useful in working through the following argument. With this in mind we make the ansatz that each eigenfunction may be written as $\mathbf{u} = \mathbf{f}_{lmn}. * \mathbf{q}$ for a nonzero constant q. To find such eigenfunctions, substitute the ansatz into equation (3.2). For each nonzero $\mathbf{p}^T = (\alpha_l, \beta_m, \gamma_n)$, $\mathbf{q}$ satisfies

$$\xi\mathbf{q} = (1 - 2\nu)(\mathbf{p}^T\mathbf{p})\mathbf{q} + (\mathbf{p}^T\mathbf{q})\mathbf{p}.$$

Note that we have the freedom to multiply q by an arbitrary nonzero scalar. Given the nonzero vector $\mathbf{p}$, $\mathbf{q}$ corresponds is an eigenvector if either $\mathbf{q}\cdot\mathbf{p} = 0$, or $\mathbf{q} = \mathbf{p}$. The two families of modes are solenoidal $\mathbf{q}\cdot\mathbf{p} = 0$, and compressive $\mathbf{q} = \mathbf{p}$.

The solenoidal eigenfunctions are spanned by the functions

$$\left\{ \begin{array}{l} \mathbf{A}_{lmn}(\mathbf{x}) = (0, \gamma_n g_{lmn}(\mathbf{x}), -\beta_m h_{lmn}(\mathbf{x}))\,, \\ \mathbf{B}_{lmn}(\mathbf{x}) = (\gamma_n f_{lmn}(\mathbf{x}), 0, -\alpha_l h_{lmn}(\mathbf{x}))\,, \\ \mathbf{C}_{lmn}(\mathbf{x}) = (\beta_m f_{lmn}(\mathbf{x}), -\alpha_l g_{lmn}(\mathbf{x}), 0)\,, \end{array} \right. \tag{4.1}$$

and have the eigenvalues

$$\xi^{(s)}_{lmn} = (1 - 2\nu)(\alpha_l^2 + \beta_m^2 + \gamma_n^2). \tag{4.2}$$

Each eigenfunction $\mathbf{A}_{lmn}$, $\mathbf{B}_{lmn}$, or $\mathbf{C}_{lmn}$ has zero divergence. If $l = 0$ then $\alpha_l = f_{lmn} = 0$. The number of solenoidal eigenfunctions corresponding to (l, m, n) (the multiplicity of $\xi^{(s)}_{lmn}$) is the number of nonzero indices minus one.

The compressive eigenfunctions take the form

$$\mathbf{D}_{lmn}(x, y, z) = (\alpha_l f_{lmn}(x, y, z), \beta_m g_{lmn}(x, y, z), \gamma_n h_{lmn}(x, y, z))\,, \tag{4.3}$$

for $(l, m, n) \neq \mathbf{0}$. have nonzero divergence and the eigenvalues

$$\xi^{(c)}_{lmn} = 2(1 - \nu)(\alpha_l^2 + \beta_m^2 + \gamma_n^2), \quad l^2 + m^2 + n^2 \neq 0 \tag{4.4}$$

Note that $\mathbf{D}_{000} = \mathbf{0}$ is omitted.

The number of eigenfunctions corresponding to (l, m, n) (and the multiplicity of $\xi^{(s)}_{lmn}$) is the number of nonzero indices. If (l, m, n) is nonzero, then $\xi^{(c)}_{lmn}$ is an eigenvalue. If two components of (l, m, n) are nonzero, then $\xi^{(s)}_{lmn}$ is an eigenvalue. If three components of (l, m, n) are nonzero, then $\xi^{(s)}_{lmn}$ is a double eigenvalue.

MATLAB scripts are available from the first author for computing some number of the lowest frequency modes. The modes corresponding to wave numbers below some wave number $(l_{\max}, m_{\max}, n_{\max})$ are computed. The smallest omitted mode is determined. Only the modes below the smallest omitted mode are returned. The maximum wave number is determined so that the whole process is relatively efficient.

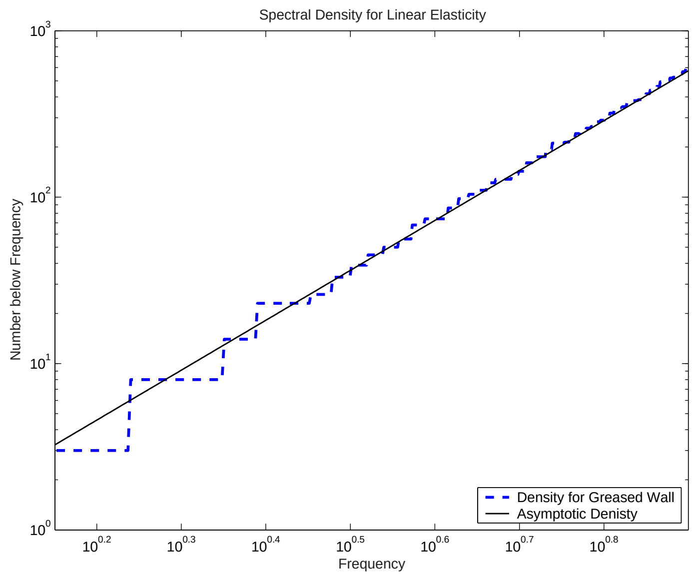

Figure 4.1. The spectral density for the linear elasticity operator over the domain $[0, \pi]^3$ with greased-wall boundary conditions (blue dashed line) is compared to the asymptotic spectral density for the domain. The material properties are $\lambda = \mu = \rho = 1$.

5 Completeness

We have located a large number of eigenvalues and eigenvectors. We will now show that we have in fact found all of them. To do this we will show that any function $\mathbf{u}(x, y, z)$ that satisfies our boundary conditions can be expanded in terms of the the eigenfunctions $\mathbf{D}_{lmn}$, $\mathbf{B}_{lmn}$, $\mathbf{A}_{lmn}$, and $\mathbf{C}_{lmn}$. The functions (ϕ_{lmn}) defined by

$$\phi_{lmn}(x, y, z) = \cos(\alpha_l x) \cos(\beta_m y) \cos(\gamma_n z),$$

appeared earlier for example in

$$\nabla \cdot \mathbf{D}_{lmn} = (\alpha_l^2 + \beta_m^2 + \gamma_n^2)\phi_{lmn}(x, y, z)$$

The functions (ϕ_{lmn}) are a complete set of eigenfunctions (of Δ over Ω with homogeneous Neumann boundary conditions). Furthermore, we are able to expand any smooth function $f(x, y, z)$ whose integral vanishes over our hexahedron using $(\phi_{lmn}(x, y, z))$ without using the term with $l = m = n = 0$. This shows that we can use the functions $\mathbf{D}_{lmn}$ to find a function that satisfies all of the boundary conditions and has the same divergence as an arbitrary function $\mathbf{u}(x, y, z)$ that satisfies our boundary conditions.

It follows that we can subtract a linear combination of the eigenvectors $\mathbf{D}_{lmn}$ from the function $\mathbf{u}(x, y, z)$ to get a function $\mathbf{u}_1(x, y, z)$ that has a zero divergence, and that satisfies all of our boundary conditions.

Using the fact that the functions f_{lmn} are complete, we can express the x component of the function $\mathbf{u}_1(x, y, z)$ in terms of $f_{lmn}(x, y, z)$. This shows that using the functions $\mathbf{B}_{lmn}$ we can eliminate all terms in the x component of $\mathbf{u}_1$ except for those that do not have any dependence on z (the terms with $n = 0$). Now using the functions $\mathbf{C}_{lmn}$ we can eliminate all terms in the x component that do not have any dependence on y. It follows that using the functions $\mathbf{D}_{lmn}$, $\mathbf{B}_{lmn}$ and $\mathbf{C}_{lmn}$, we can subtract off terms from an arbitrary function $\mathbf{u}$ that satisfies our boundary conditions to get a function $\mathbf{u}_2(x, y, z)$ that has no divergence, satisfies all of our boundary conditions, and whose x component is only a function of x.

Similar arguments show that we can use the functions $\mathbf{A}_{lmn}$ to get a function $\mathbf{u}_3(x, y, z)$ that has a zero divergence, satisfies all of the boundary conditions, whose x component only depends on x, and whose y component only depends on x and y. The function has the form

$$\mathbf{u}_3(x, y, z) = (f(x), g(x, y), w(x, y, z)) \,.$$

In order for this to have a zero divergence we must have

$$w(x, y, z) = -z(f'(x) + \frac{\partial}{\partial y} g(x, y)) + \psi(x, y).$$

Since $w = 0$ at $z = 0$, we must have $\psi(x, y) = 0$. Since $w = 0$ at $z = L_z$, we must have $f'(x) + \frac{\partial}{\partial y} g(x, y) = 0$. This implies that

$$g(x, y) = -yf'(x) + q(x)$$

In order to satisfy the boundary conditions at $y = 0, L_y$ this requires that $f'(x) = 0$, $q(x) = 0$, and hence $g(x, y) = 0$. It follows that $f(x) = constant$. In order to satisfy the boundary conditions we must have $f = 0$. It follows that $\mathbf{u}_3 = 0$. We see that using our eigenfunctions we have been able to expand an arbitrary function $\mathbf{u}$ that satisfies our boundary conditions. This shows that we have in fact found all of the eigenfunctions.

References

[1] H. P. Baltes and E. R. Wilf. *Spectra of finite systems*. Bibliographisches Institut, Mannheim, 1976.

[2] R.D. Blevins. *Formulas for natural frequency and mode shape*. Van Nostrand Reinhold Co., New York, 1979.

[3] P. G. Ciarlet. *Lectures on three-dimensional elasticity*. Tata institute lectures on mathematics v 71. Tata institute of fundamental research, 1983.

[4] M. Dupuis, R. Mazo, and L. Onsager. Surface specific heat of an isotropic solid at low temperatures. *J. Chem. Phys.*, 33(5):1452–1455, 1960.

[5] W. Findley, J. Lai, and K. Onaran. *Creep and relaxation of nonlinear viscoelastic materials*. Dover, 1976.

[6] L. D. Landau and E. M. Lifshitz. *Theory of elasticity*. Course of theoretical physics volume 7. Pergamon Press, 1970.