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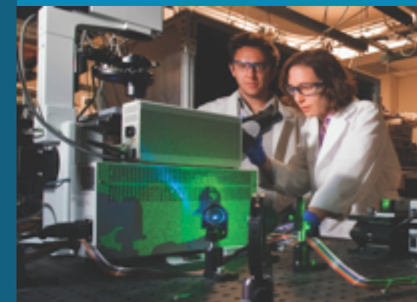
Explicit Partitioned Methods for Interface Coupling



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Sandia National Laboratories

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PARTITIONED METHODS

Complex applications often require coupling of different codes or separately meshed regions through non-matching interfaces

Applications:

- Solid-solid interaction
- Transmission problems
- Fluid-structure interaction

Interface Conditions

$$u_1 = u_2 \text{ on } \gamma$$
$$F_1 = -F_2 \text{ on } \gamma$$

Model 1

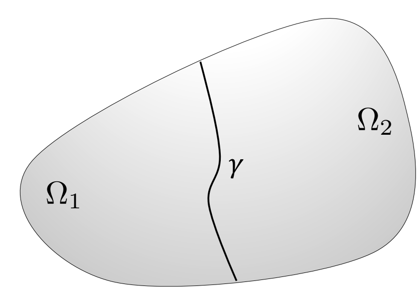
$$\dot{u}_1 + \mathcal{L}_1 u_1 = f_1 \text{ in } \Omega_1$$

Model 2

$$\dot{u}_2 + \mathcal{L}_2 u_2 = f_2 \text{ in } \Omega_2$$

We seek a method that:

- Enables independent solution of each subdomain problem
- Is explicit and non-iterative for time-dependent governing equations
- Is accurate and stable for interfaces with non-matching grids

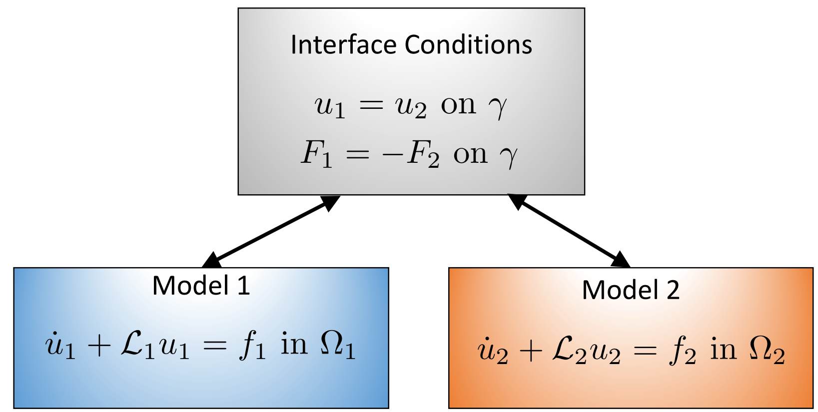


PARTITIONED METHODS FOR INTERFACE COUPLING

Existing approaches

Direct estimation of flux/force

- Subdomain solution used to approximate flux and project to other side
- Mathematically equivalent to single step of iterative method
- Can lead to stability or accuracy problems
- Jaiman et al. 2005; Dryja and Widlund 1998;



Lagrange multiplier approach

- Lagrange multiplier is force/flux
- Derived from monolithic formulation
- Not compatible with explicit time integration
- Mortar (Bernardi et al. 1994); FETI (Farhat and Roux 1991); Other (Zheng et al. 2007; Ross et al. 2009)

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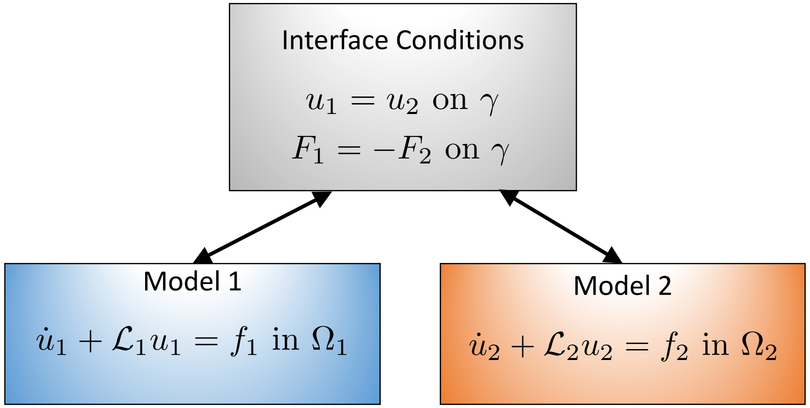

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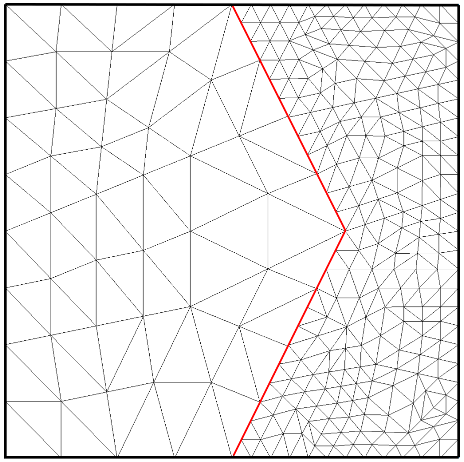
We present a new partitioned method derived from a well-posed monolithic mixed formulation of the coupled problem that is stable, accurate, and compatible with explicit time stepping

MODEL PROBLEMS

- Two domains connected by an interface
- Non-matching grids, but spatially coincident
- Same governing equations in each domain



Linear solid-solid interaction

$$\ddot{\boldsymbol{u}}_i - \nabla \cdot \boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \boldsymbol{f}_i \text{ in } \Omega_i \times [0, T]$$

$$\boldsymbol{u}_i = \boldsymbol{g}_i \text{ on } \Gamma_i \times [0, T]$$

$$\boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \lambda_i(\nabla \cdot \boldsymbol{u}_i)I + 2\mu_i\boldsymbol{\varepsilon}_i(\boldsymbol{u}_i)$$

Coupling Conditions

$$\boldsymbol{u}_1(\boldsymbol{x}, t) = \boldsymbol{u}_2(\boldsymbol{x}, t) \text{ and } \boldsymbol{\sigma}_1(\boldsymbol{x}, t) \cdot \boldsymbol{n}_\gamma = \boldsymbol{\sigma}_2(\boldsymbol{x}, t) \cdot \boldsymbol{n}_\gamma \text{ on } \gamma \times [0, T]$$

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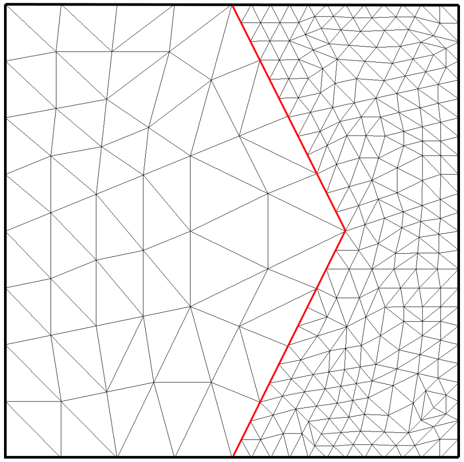

Advection-Diffusion Transmission Problem

$$\dot{\varphi}_i - \nabla \cdot F_i(\varphi_i) = f_i \text{ in } \Omega_i \times [0, T]$$

$$\varphi_i = g_i \text{ in } \Gamma_i \times [0, T]$$

$$F_i(\varphi_i) = \epsilon_i \nabla \varphi_i - \boldsymbol{u}_i \varphi_i$$

Coupling Conditions

$$\varphi_1(\boldsymbol{x}, t) = \varphi_2(\boldsymbol{x}, t) \text{ and } F_1(\boldsymbol{x}, t) \cdot \boldsymbol{n}_\gamma = F_2(\boldsymbol{x}, t) \cdot \boldsymbol{n}_\gamma \text{ on } \gamma \times [0, T]$$

MONOLITHIC FORMULATION LINEAR ELASTICITY

Mixed Formulation

$$\begin{aligned} \ddot{\boldsymbol{u}}_1 - \nabla\cdot\boldsymbol{\sigma}_1(\boldsymbol{u}_1) &= \boldsymbol{f}_1 \text{ in } \Omega_1 \\ \boldsymbol{\sigma}_1\cdot\boldsymbol{n}_1 &= -\boldsymbol{t} \text{ on } \gamma \\ \ddot{\boldsymbol{u}}_2 - \nabla\cdot\boldsymbol{\sigma}_2(\boldsymbol{u}_2) &= \boldsymbol{f}_2 \text{ in } \Omega_2 \\ \boldsymbol{\sigma}_2\cdot\boldsymbol{n}_2 &= \boldsymbol{t} \text{ on } \gamma \\ \boldsymbol{u}_1 &= \boldsymbol{u}_2 \text{ on } \gamma \end{aligned}$$

Discretize

$$\begin{aligned} \boldsymbol{u}_1 &\in S_1^h \subset H^1_{\Gamma_1}(\Omega_1) \\ \boldsymbol{u}_2 &\in S_2^h \subset H^1_{\Gamma_2}(\Omega_2) \\ \boldsymbol{t} &\in G_\gamma^h \subset H^{-1/2}(\gamma) \end{aligned}$$

Semi-Discrete System

$$\begin{aligned} M_1\ddot{\mathbf{u}}_1 + G_1^T\mathbf{t} &= \mathbf{f}_1(\mathbf{u}_1) \\ M_2\ddot{\mathbf{u}}_2 - G_2^T\mathbf{t} &= \mathbf{f}_2(\mathbf{u}_2) \\ G_1\mathbf{u}_1 - G_2\mathbf{u}_2 &= 0 \end{aligned}$$

Mass matrix $(M_i)_{kl} = (N_{i,k}, N_{i,l})_\Omega$

Coupling matrix $(G_i)_{kl} = (N_{i,k}, \nu_l)_\gamma$

Force vector $\mathbf{f}_{i,k} = -(\nabla N_{i,k}, \boldsymbol{\sigma}_i)_\Omega + (N_{i,k}, \boldsymbol{f}_i)_\Omega$

- System of 3 equations for 3 unknowns
- Neumann boundary conditions involve unknown contact force $\mathbf{t}$
- Contact force continuity subsumed into the equations
- Displacement discontinuity enforced explicitly

LAGRANGE MULTIPLIER FORMULATION

- Index 2 Differential Algebraic Equation (DAE)
- Lagrange multiplier is not an implicit function of displacements
- Not compatible with explicit treatment of interface force (t)

$$M_1 \ddot{\mathbf{u}}_1 + G_1^T \mathbf{t} = \mathbf{f}_1(\mathbf{u}_1)$$

$$M_2 \ddot{\mathbf{u}}_2 - G_2^T \mathbf{t} = \mathbf{f}_2(\mathbf{u}_2)$$

$$G_1 \mathbf{u}_1 - G_2 \mathbf{u}_2 = 0$$

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$$G_1 \mathbf{u}_1 - G_2 \mathbf{u}_2 = 0$$

Conversion
To index 1 DAE

- Replace displacement continuity constraint on the interface with acceleration continuity constraint
- Under suitable assumptions new constraint implies the original $(\dot{\mathbf{u}}_{1,\gamma}(0) = \dot{\mathbf{u}}_{2,\gamma}(0), \quad \mathbf{u}_{1,\gamma}(0) = \mathbf{u}_{2,\gamma}(0))$
- New system enables a fully explicit treatment of t

$$M_1 \ddot{\mathbf{u}}_1 + G_1^T \mathbf{t} = \mathbf{f}_1(\mathbf{u}_1)$$
$$M_2 \ddot{\mathbf{u}}_2 - G_2^T \mathbf{t} = \mathbf{f}_2(\mathbf{u}_2)$$
$$G_1 \ddot{\mathbf{u}}_1 - G_2 \ddot{\mathbf{u}}_2 = 0$$

SOLVING FOR THE INTERFACE FORCE

Linear system of equations

$$\left[\begin{array}{ccc:cc} M_{1,\gamma} & 0 & G_1^T & 0 & 0 \\ 0 & M_{2,\gamma} & -G_2^T & 0 & 0 \\ G_1 & -G_2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & M_{1,0} & 0 \\ 0 & 0 & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{2,\gamma} \\ \mathbf{t} \\ \hdashline \ddot{\mathbf{u}}_{1,0} \\ \ddot{\mathbf{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) \\ \mathbf{f}_{2,\gamma}(\mathbf{u}_2) \\ 0 \\ \hdashline \mathbf{f}_{1,0}(\mathbf{u}_1) \\ \mathbf{f}_{2,0}(\mathbf{u}_2) \end{array}\right]$$

For lumped mass matrices, can directly separate the interface block

$$\begin{bmatrix} M_{1,\gamma} & 0 & G_1^T \\ 0 & M_{2,\gamma} & -G_2^T \\ G_1 & -G_2 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{2,\gamma} \\ \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) \\ \mathbf{f}_{2,\gamma}(\mathbf{u}_2) \\ \mathbf{0} \end{bmatrix}$$

And solve for the interface force

$$\left(G_1 M_{1,\gamma}^{-1} G_1^T + G_2 M_{2,\gamma}^{-1} G_2^T\right)\mathbf{t} = \left(G_1 M_{1,\gamma}^{-1}\mathbf{f}_{1,\gamma}(\mathbf{u}_1) - G_2 M_{2,\gamma}^{-1}\mathbf{f}_{2,\gamma}(\mathbf{u}_2)\right)$$

Note: Solvability requires G_1 and G_2 to have full column ranks

TIME DISCRETIZATION

Given interface force (t) as a function of displacements, we have the underlying system of ODEs

$$\left[\begin{array}{cc:cc} M_{1,\gamma} & 0 & 0 & 0 \\ 0 & M_{1,0} & 0 & 0 \\ \hdashline 0 & 0 & M_{2,\gamma} & 0 \\ 0 & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{1,0} \\ \hdashline \ddot{\mathbf{u}}_{2,\gamma} \\ \ddot{\mathbf{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) - G_1^T \mathbf{t}(\mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{f}_{1,0}(\mathbf{u}_1) \\ \hdashline \mathbf{f}_{2,\gamma}(\mathbf{u}_2) + G_2^T \mathbf{t}(\mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{f}_{2,0}(\mathbf{u}_2) \end{array}\right]$$

Discretize in time

$$\ddot{D}^{n+1}(\mathbf{u}_i) = (\mathbf{u}_i^{n+1} - 2\mathbf{u}_i^n + \mathbf{u}_i^{n-1})/\Delta t^2$$

$$\left[\begin{array}{cc} M_{i,\gamma} & 0 \\ 0 & M_{i,0} \end{array}\right] \left[\begin{array}{c} \ddot{D}^{n+1}\mathbf{u}_{i,\gamma} \\ \ddot{D}^{n+1}\mathbf{u}_{i,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{i,\gamma}^n + (-1)^i G_i^T \mathbf{t}^n \\ \mathbf{f}_{i,0} \end{array}\right]$$

- Time discretization both discretizes the system in time and partitions the subdomain equations
- As long as time step is within the stability region of the time integrator, the partitioned scheme is stable
- Not subject to splitting errors characteristic of iterative partitioned methods

IVR ALGORITHM

In this formulation, the Lagrange multiplier can be expressed as an implicit function of the displacements.

We refer to this scheme as Implicit Value Recovery (IVR)

1. Compute the force vectors

$$\mathbf{f}_i^n := \mathbf{f}_i(\mathbf{u}_i^n)$$

2. Estimate interface boundary condition

$$\left(G_1 M_1^{-1} G_1^T + G_2 M_2^{-1} G_2^T\right) \mathbf{t}^n = \left(G_1 M_1^{-1} \mathbf{f}_{1,\gamma}^n - G_2 M_2^{-1} \mathbf{f}_{2,\gamma}^n\right)$$

3. Update solution

$$\begin{bmatrix} M_{i,\gamma} & M_{i,\gamma 0} \\ M_{i,0\gamma} & M_{i,0} \end{bmatrix} \begin{bmatrix} \ddot{D}^{n+1}\mathbf{u}_{i,\gamma} \\ \ddot{D}^{n+1}\mathbf{u}_{i,0} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{i,\gamma}^n + (-1)^i G_i^T \mathbf{t}^n \\ \mathbf{f}_{i,0}^n \end{bmatrix}$$

(Peterson, Bochev, Kuberry 2018)

MONOLITHIC FORMULATION ADVECTION-DIFFUSION

Mixed Formulation

$$\dot{\varphi}_1 - \nabla \cdot F_1(\varphi_1) = f_1 \text{ in } \Omega_1$$
$$F_1 \cdot \boldsymbol{n}_1 = -\lambda \text{ on } \gamma$$
$$\ddot{\varphi}_2 - \nabla \cdot F_2(\varphi_2) = f_2 \text{ in } \Omega_2$$
$$F_2 \cdot \boldsymbol{n}_2 = \lambda \text{ on } \gamma$$
$$\varphi_1 = \varphi_2 \text{ on } \gamma$$

Discretize

$$\varphi_1 \in S_1^h \subset H^1_{\Gamma_1}(\Omega_1)$$
$$\varphi_2 \in S_2^h \subset H^1_{\Gamma_2}(\Omega_2)$$
$$\lambda \in G_\gamma^h \subset H^{-1/2}(\gamma)$$

Semi-Discrete System

$$M_1 \dot{\boldsymbol{\varphi}}_1 + G_1^{\mathsf{T}} \lambda = \mathbf{f}_1(\boldsymbol{\varphi}_1)$$
$$M_2 \dot{\boldsymbol{\varphi}}_2 - G_2^{\mathsf{T}} \lambda = \mathbf{f}_2(\boldsymbol{\varphi}_2)$$
$$G_1 \boldsymbol{\varphi}_1 - G_2 \boldsymbol{\varphi}_2 = 0$$

Mass matrix $(M_i)_{kl} = (N_{i,k}, N_{i,l})_\Omega$

Coupling matrix $(G_i)_{kl} = (N_{i,k}, \nu_l)_\gamma$

Force vector $\mathbf{f}_{i,k} = -(\nabla N_{i,k}, F_i)_\Omega + (N_{i,k}, f_i)_\Omega$

MONOLITHIC MIXED FORMULATION

Mixed Formulation

$$\begin{aligned} \dot{\varphi}_1 - \nabla \cdot F_1(\varphi_1) &= f_1 \text{ in } \Omega_1 \\ F_1 \cdot \boldsymbol{n}_1 &= -\lambda \text{ on } \gamma \\ \ddot{\varphi}_2 - \nabla \cdot F_2(\varphi_2) &= f_2 \text{ in } \Omega_2 \\ F_2 \cdot \boldsymbol{n}_2 &= \lambda \text{ on } \gamma \\ \varphi_1 &= \varphi_2 \text{ on } \gamma \end{aligned}$$

Discretize

$$\varphi_1 \in S_1^h \subset H^1_{\Gamma_1}(\Omega_1)$$

$$\varphi_2 \in S_2^h \subset H^1_{\Gamma_2}(\Omega_2)$$

$$\lambda \in G_\gamma^h \subset H^{-1/2}(\gamma)$$

Semi-Discrete System

$$M_1\dot{\boldsymbol{\varphi}}_1 + G_1^{\mathsf{T}}\lambda = \mathbf{f}_1(\boldsymbol{\varphi}_1)$$

$$M_2\dot{\boldsymbol{\varphi}}_2 - G_2^{\mathsf{T}}\lambda = \mathbf{f}_2(\boldsymbol{\varphi}_2)$$

$$G_1\boldsymbol{\varphi}_1 - G_2\boldsymbol{\varphi}_2 = 0$$

Mass matrix $(M_i)_{kl} = (N_{i,k}, N_{i,l})_\Omega$

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Conversion To index 1 DAE

$$M_1\dot{\boldsymbol{\varphi}}_1 + G_1^{\mathsf{T}}\lambda = \mathbf{f}_1(\boldsymbol{\varphi}_1)$$

$$M_2\dot{\boldsymbol{\varphi}}_2 - G_2^{\mathsf{T}}\lambda = \mathbf{f}_2(\boldsymbol{\varphi}_2)$$

$$G_1\dot{\boldsymbol{\varphi}}_1 - G_2\dot{\boldsymbol{\varphi}}_2 = 0$$

$$\begin{bmatrix} M_1 & 0 & G_1^T \\ 0 & M_2 & -G_2^T \\ G_1 & -G_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\varphi}}_1 \\ \dot{\boldsymbol{\varphi}}_2 \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1(\boldsymbol{\varphi}_1) \\ \mathbf{f}_2(\boldsymbol{\varphi}_2) \\ 0 \end{bmatrix}$$

SOLVING FOR THE INTERFACE FLUX

Linear system of equations

$$\left[\begin{array}{ccc|cc} M_{1,\gamma} & 0 & G_1^T & M_{1,\gamma 0} & 0 \\ 0 & M_{2,\gamma} & -G_2^T & 0 & M_{2,\gamma 0} \\ G_1 & -G_2 & 0 & 0 & 0 \\ \hline M_{1,0\gamma} & 0 & 0 & M_{1,0} & 0 \\ 0 & M_{2,0\gamma} & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \dot{\boldsymbol{\varphi}}_{1,\gamma} \\ \dot{\boldsymbol{\varphi}}_{2,\gamma} \\ \boldsymbol{\lambda} \\ \hline \dot{\boldsymbol{\varphi}}_{1,0} \\ \dot{\boldsymbol{\varphi}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\boldsymbol{\varphi}_1) \\ \mathbf{f}_{2,\gamma}(\boldsymbol{\varphi}_2) \\ 0 \\ \hline \mathbf{f}_{1,0}(\boldsymbol{\varphi}_1) \\ \mathbf{f}_{2,0}(\boldsymbol{\varphi}_2) \end{array}\right]$$

For consistent mass matrices, the interface system is

$$\begin{bmatrix} A_{1,\gamma} & 0 & G_1^T \\ 0 & A_{2,\gamma} & -G_2^T \\ G_1 & -G_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\varphi}}_{1,\gamma} \\ \dot{\boldsymbol{\varphi}}_{2,\gamma} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{f}}_{1,\gamma}(\boldsymbol{\varphi}_1) \\ \widehat{\mathbf{f}}_{2,\gamma}(\boldsymbol{\varphi}_2) \\ \mathbf{0} \end{bmatrix}$$

where

$$A_i = M_{i,\gamma} - M_{i,\gamma 0} M_{i,0}^{-1} M_{i,0\gamma}$$

$$\widehat{\mathbf{f}}_{i,\gamma}(\boldsymbol{\varphi}_i) = \mathbf{f}_{i,\gamma}(\boldsymbol{\varphi}_i) - M_{i,\gamma 0} M_{i,0}^{-1} \mathbf{f}_{i,0}(\boldsymbol{\varphi}_i)$$

And the interface flux

$$\left(G_1 A_{1,\gamma}^{-1} G_1^T + G_2 A_{2,\gamma}^{-1}\right) \boldsymbol{\lambda} = \Big(G_1 A_{1,\gamma}^{-1} \widehat{\mathbf{f}}_{1,\gamma}(\boldsymbol{\varphi}_1) - G_1 A_{2,\gamma}^{-1} \widehat{\mathbf{f}}_{2,\gamma}(\boldsymbol{\varphi}_2)\Big)$$

TIME DISCRETIZATION

Given interface flux as a function of subdomain state, we have the underlying system of ODEs

$$\left[\begin{array}{cc:cc} M_{1,\gamma} & M_{1,\gamma 0} & 0 & 0 \\ M_{1,0\gamma} & M_{1,0} & 0 & 0 \\ \hdashline 0 & 0 & M_{2,\gamma} & M_{2,\gamma 0} \\ 0 & 0 & M_{2,0\gamma} & M_{2,0} \end{array}\right] \left[\begin{array}{c} \dot{\boldsymbol{\varphi}}_{1,\gamma} \\ \dot{\boldsymbol{\varphi}}_{1,0} \\ \hdashline \dot{\boldsymbol{\varphi}}_{2,\gamma} \\ \dot{\boldsymbol{\varphi}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\boldsymbol{\varphi}_1) - G_1^T \boldsymbol{\lambda}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \\ \mathbf{f}_{1,0}(\boldsymbol{u}_1) \\ \hdashline \mathbf{f}_{2,\gamma}(\boldsymbol{\varphi}_2) + G_2^T \boldsymbol{\lambda}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \\ \mathbf{f}_{2,0}(\boldsymbol{\varphi}_2) \end{array}\right]$$

Discretize in time and partition the subdomain equations

$$\dot{D}^{n+1} \boldsymbol{\varphi}_i = (\boldsymbol{\varphi}_i^{n+1} - \boldsymbol{\varphi}_i^n)/\Delta t$$

$$\left[\begin{array}{cc} M_{i,\gamma} & M_{i,\gamma 0} \\ M_{i,0\gamma} & M_{i,0} \end{array}\right] \left[\begin{array}{c} \dot{D}^{n+1} \boldsymbol{\varphi}_{i,\gamma} \\ \dot{D}^{n+1} \boldsymbol{\varphi}_{i,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{i,\gamma}^n + (-1)^i G_i^T \boldsymbol{\lambda}^n \\ \mathbf{f}_{i,0}^n \end{array}\right]$$

IVR ALGORITHM – CONSISTENT MASS MATRICES

1a. Compute the force vectors

$$\mathbf{f}_i^n = \mathbf{f}_i(\boldsymbol{\varphi}_i^n)$$

1b. Compute generalized mass matrices and generalized force vectors

$$A_i = M_{i,\gamma} - M_{i,\gamma 0} M_{i,0}^{-1} M_{i,0\gamma} \qquad \widehat{\mathbf{f}}_{i,\gamma}(\boldsymbol{\varphi}_i) = \mathbf{f}_{i,\gamma}(\boldsymbol{\varphi}_i) - M_{i,\gamma 0} M_{i,0}^{-1} \mathbf{f}_{i,0}(\boldsymbol{\varphi}_i)$$

2. Estimate interface boundary condition

$$\left(G_1 A_{1,\gamma}^{-1} G_1^T + G_2 A_{2,\gamma}^{-1}\right) \boldsymbol{\lambda} = \left(G_1 A_{1,\gamma}^{-1} \widehat{\mathbf{f}}_{1,\gamma}(\boldsymbol{\varphi}_1) - G_1 A_{2,\gamma}^{-1} \widehat{\mathbf{f}}_{2,\gamma}(\boldsymbol{\varphi}_2)\right)$$

3. Update solution

$$\begin{bmatrix} M_{i,\gamma} & M_{i,\gamma 0} \\ M_{i,0\gamma} & M_{i,0} \end{bmatrix} \begin{bmatrix} \dot{D}^{n+1} \boldsymbol{\varphi}_{i,\gamma} \\ \dot{D}^{n+1} \boldsymbol{\varphi}_{i,0} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{i,\gamma}^n + (-1)^i G_i^T \boldsymbol{\lambda}^n \\ \mathbf{f}_{i,0}^n \end{bmatrix}$$

RELATIONSHIP TO OTHER METHODS

$$\left(G_1 M_{1,\gamma}^{-1} G_1^T + G_2 M_{2,\gamma}^{-1} G_2^T\right) \mathbf{t} = \left(G_1 M_{1,\gamma}^{-1} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) - G_2 M_{2,\gamma}^{-1} \mathbf{f}_{2,\gamma}(\mathbf{u}_2)\right)$$

Other methods lead to similar linear systems

- Hybrid methods (Brezzi and Fortin 1991; Cockburn et al. 2009) and FETI methods (Farhat and Roux 1991; Farhat et al. 1994)

$$\left(G_1 K_1^{-1} G_1^T + G_2 K_2^{-1} G_2^T\right) \mathbf{t} = \left(G_1 K_1^{-1} \mathbf{f}_1 - G_2 K_2^{-1} \mathbf{f}_2\right)$$

- These systems involve Schur complements of subdomain stiffness matrices rather than subdomain mass matrices
- Typically used for domain decomposition rather than partitioning
- Solve for Lagrange multipliers then back solve for the displacements

WELL-POSEDNESS OF IVR FORMULATION

Consider semi-discrete equations for the linear elastic case

$$\boxed{\begin{aligned} &M_1\ddot{\mathbf{u}}_1 + G_1^T\mathbf{t} = \mathbf{f}_1(\mathbf{u}_1) \\ &M_2\ddot{\mathbf{u}}_2 - G_2^T\mathbf{t} = \mathbf{f}_2(\mathbf{u}_2) \\ &G_1\ddot{\mathbf{u}}_1 - G_2\ddot{\mathbf{u}}_2 = 0 \end{aligned}}$$

We can denote a mixed bilinear form corresponding to the terms on the left hand side as

$$B(\ddot{\boldsymbol{u}}_1^h, \ddot{\boldsymbol{u}}_2^h, \boldsymbol{t}^h; \boldsymbol{v}_1^h, \boldsymbol{v}_2^h, \boldsymbol{s}^h) := a(\ddot{\boldsymbol{u}}_1^h, \ddot{\boldsymbol{u}}_2^h; \boldsymbol{v}_1^h, \boldsymbol{v}_2^h) + b(\boldsymbol{v}_1^h, \boldsymbol{v}_2^h; \boldsymbol{t}^h) + b(\ddot{\boldsymbol{u}}_1^h, \ddot{\boldsymbol{u}}_2^h; \boldsymbol{s}^h)$$

where

$$a(\ddot{\boldsymbol{u}}_1^h, \ddot{\boldsymbol{u}}_2^h; \boldsymbol{v}_1^h, \boldsymbol{v}_2^h) = (\ddot{\boldsymbol{u}}_1^h, \boldsymbol{v}_1^h)_{0,\Omega_1} + (\ddot{\boldsymbol{u}}_2^h, \boldsymbol{v}_2^h)_{0,\Omega_2}$$

$$b(\boldsymbol{v}_1^h, \boldsymbol{v}_2^h; \boldsymbol{t}^h) = (\boldsymbol{v}_1^h - \boldsymbol{v}_2^h, \boldsymbol{t}^h)_{0,\gamma}$$

$$\{\boldsymbol{u}_1^h, \boldsymbol{u}_2^h, \boldsymbol{t}^h\} \in S_{1,\Gamma}^h \times S_{2,\Gamma}^h \times G_\gamma^h \qquad \{\boldsymbol{v}_1^h, \boldsymbol{v}_2^h, \boldsymbol{s}^h\} \in S_{1,\Gamma}^h \times S_{2,\Gamma}^h \times G_\gamma^h$$

WELL-POSEDNESS OF IVR FORMULATION

Well-posedness of $B(\cdot,\cdot)$ requires $a(\cdot,\cdot)$ to be coercive and $b(\cdot,\cdot)$ to satisfy an inf-sup conditions

Can easily show that $a(\cdot,\cdot)$ is coercive

$$a(\boldsymbol{v}_1^h,\boldsymbol{v}_2^h;\boldsymbol{v}_1^h,\boldsymbol{v}_2^h) = \|\boldsymbol{v}_1^h\|_{0,\Omega_1}^2 + \|\boldsymbol{v}_2^h\|_{0,\Omega_2}^2 = |||\boldsymbol{v}_1^h;\boldsymbol{v}_2^h|||^2$$

With the following conditions on the Lagrange multiplier space, can show that $b(\cdot,\cdot)$ satisfies the inf-sup condition

There exists an operator $Q : G_\gamma^h \mapsto S_{1,\Gamma}^h \times S_{2,\Gamma}^h$, such that

$$\|\boldsymbol{s}^h\|_{0,\gamma} \le C_1 \left(\boldsymbol{s}^h, (Q\boldsymbol{s}^h)_1 - (Q\boldsymbol{s}^h)_2\right)_{0,\gamma} \quad \forall \boldsymbol{s}^h \in G_\gamma^h, \quad \text{and} \tag{1}$$

$$|||Q(\boldsymbol{s}^h)||| \le C_2 h_\gamma^\alpha \|\boldsymbol{s}\|_{0,\gamma}, \quad \alpha \ge 0 \tag{2}$$

where $|||\cdot|||^2 = \|\cdot\|_{0,\Omega_1}^2 + \|\cdot\|_{0,\Omega_2}^2$ and C_1, C_2 are mesh-independent constants.

WELL-POSEDNESS OF IVR FORMULATION

Lemma: Assume that G_γ^h satisfies conditions (1) and (2), then $b(\cdot,\cdot)$ satisfies the inf-sup condition

$$\sup_{\{\boldsymbol{v}_1^h;\boldsymbol{v}_2^h\}\in X\times X} \frac{b(\boldsymbol{v}_1^h,\boldsymbol{v}_2^h;\boldsymbol{s}^h)}{|||\boldsymbol{v}_1^h;\boldsymbol{v}_2^h|||} \geq \beta\|\boldsymbol{s}^h\|_Y$$

Therefore, $B(\cdot,\cdot)$ is weakly coercive with a mesh-independent constant

Corollary: Assume that G_γ^h satisfies conditions (1) and (2), then the matrix in (3) is uniformly bounded and its inverse is uniformly bounded in the mesh size. Furthermore, the block matrix $G^T = (G_1^T, G_2^T)$ has full column rank.

$$\left[\begin{array}{ccc:cc} M_{1,\gamma} & 0 & G_1^T & 0 & 0 \\ 0 & M_{2,\gamma} & -G_2^T & 0 & 0 \\ G_1 & -G_2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & M_{1,0} & 0 \\ 0 & 0 & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{2,\gamma} \\ \mathbf{t} \\ \hdashline \ddot{\mathbf{u}}_{1,0} \\ \ddot{\mathbf{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) \\ \mathbf{f}_{2,\gamma}(\mathbf{u}_2) \\ 0 \\ \hdashline \mathbf{f}_{1,0}(\mathbf{u}_1) \\ \mathbf{f}_{2,0}(\mathbf{u}_2) \end{array}\right] \qquad (3)$$

S is invertible

$$S = G_1 M_1^{-1} G_1^T + G_2 M_2^{-1} G_2^T$$

LAGRANGE MULTIPLIER SPACE

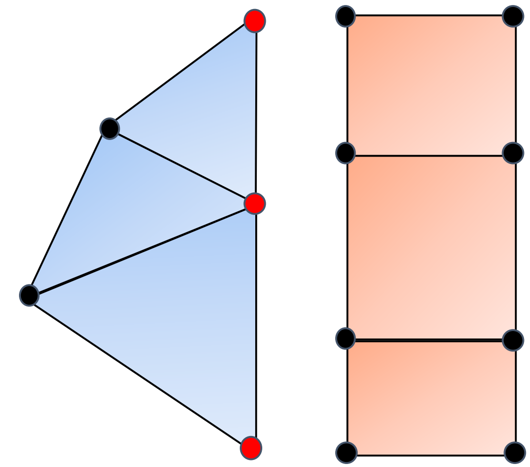
- Following mortar method approach, choose one of the interface partitions to define the Lagrange multiplier space
- For this choice we can always find an operator Q to satisfy conditions (1) and (2)
- Results in a formulation that satisfies the inf-sup condition
- Additionally, we can show that for this choice the condition number of S is bounded

Assume that $h_1 \leq h_2$ and let $\rho = h_2/h_1 > 1$.

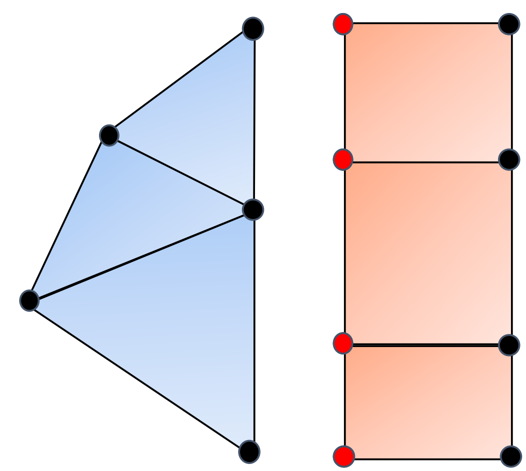
$$\text{If } G_\gamma^h = G_1^h \quad \kappa(\tilde{S}) \leq C\rho^{d-1}$$

$$\text{If } G_\gamma^h = G_2^h \quad \kappa(\tilde{S}) \leq C\rho^{d}$$

IVR(1)



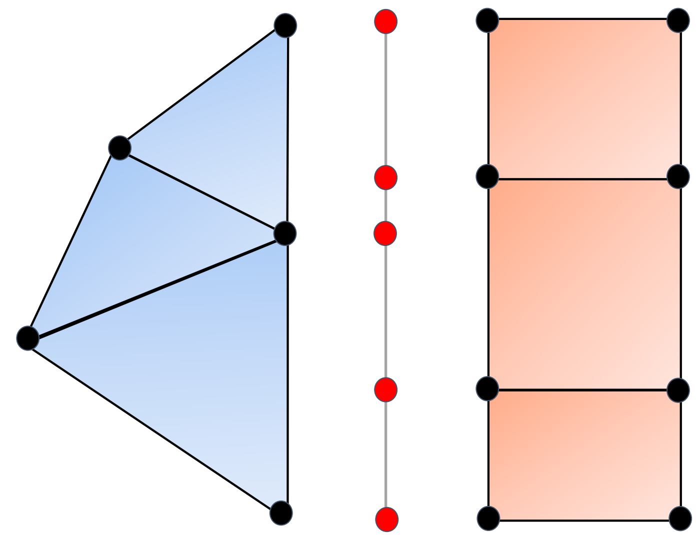
IVR(2)



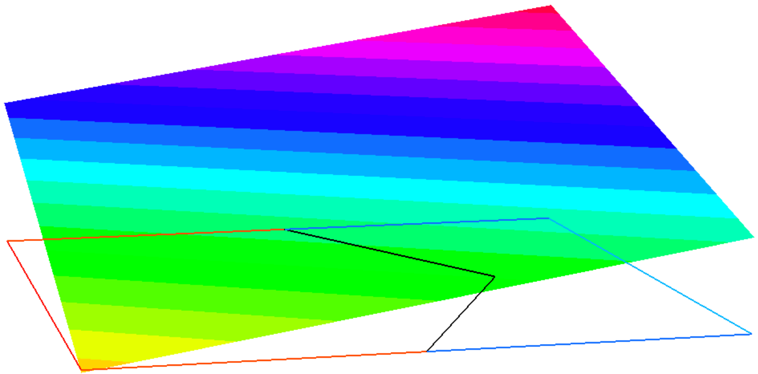
ALTERNATIVE LAGRANGE MULTIPLIER SPACE

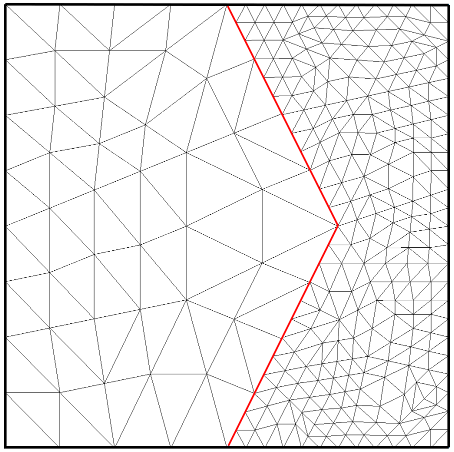
- Drawback of using the previous Lagrange multiplier spaces is consistency errors
- Expect to converge optimally, but not pass a patch test
- As an alternative, considered the use of a common refinement to define the Lagrange multiplier space
- No proof of inf-sup stability for this space, but numerical results indicate that it preserves linear fields
- Practical challenges include floating point errors and mesh quality for certain configurations

IVR(12)



LSSI PATCH TEST





Linear Elasticity

$$\ddot{\boldsymbol{u}}_i - \nabla \cdot \boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \boldsymbol{f}_i$$

$$\boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \lambda_i(\nabla \cdot \boldsymbol{u}_i)I + 2\mu_i \boldsymbol{\varepsilon}_i(\boldsymbol{u}_i)$$

$$\lambda_i = 400 \quad \text{and} \quad \mu_i = 400$$

Manufactured Solution

$$\boldsymbol{u}(\boldsymbol{x}) = (3x + 5y, 8x - 4.3y)$$

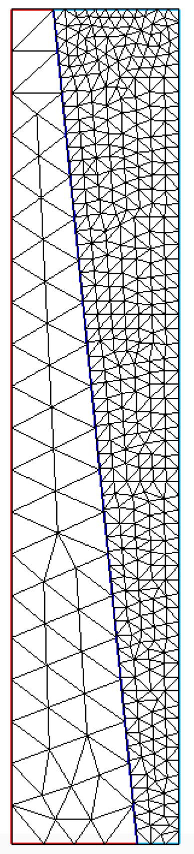
Error Norm	IVR(L1)	IVR(L2)	IVR(L12)
$L^2(0,T;L^2(\Omega))$	5.166e-04	1.468e-06	2.223e-15
$L^2(0,T;H^1(\Omega))$	1.683e-02	2.679e-05	3.412e-14

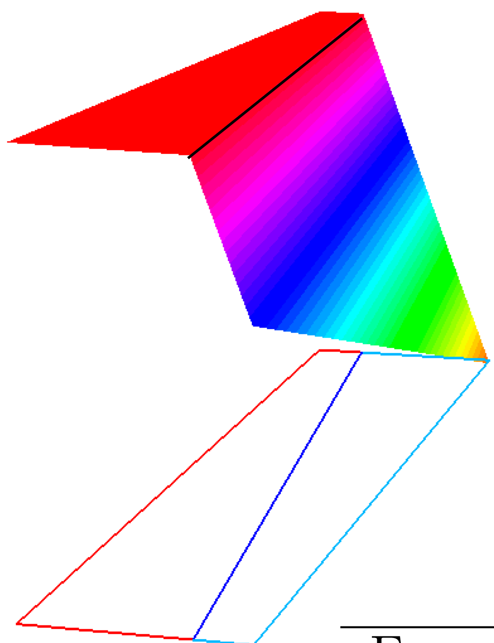
LSSI DISCONTINUOUS PATCH TEST

Manufactured Solution

$$\boldsymbol{u}_1(\boldsymbol{x}) = \left(\frac{-0.9 + x + 0.1y}{0.15}, \frac{18 - 20x - 2y}{0.15}\right)$$

$$\boldsymbol{u}_2(\boldsymbol{x}) = \left(100\left(\frac{-0.9 + x + 0.1y}{0.15}\right) - 99), 100\left(\frac{18 - 20x - 2y}{0.15}\right) + 1980\right)$$





$$\ddot{\boldsymbol{u}}_i - \nabla \cdot \boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \boldsymbol{f}_i$$

$$\boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \lambda_i(\nabla \cdot \boldsymbol{u}_i)I + 2\mu_i\boldsymbol{\varepsilon}_i(\boldsymbol{u}_i)$$

$$\lambda_1 = 40 \quad \text{and} \quad \mu_1 = 40$$

$$\lambda_2 = 0.4 \quad \text{and} \quad \mu_2 = 0.4$$

Error Norm	IVR(L1)	IVR(L2)	IVR(L12)
$L^2(0,T;L^2(\Omega))$	1.093e-04	5.418e-07	4.832e-13
$L^2(0,T;H^1(\Omega))$	4.591e-02	1.083e-04	6.658e-11

LSSI CONVERGENCE

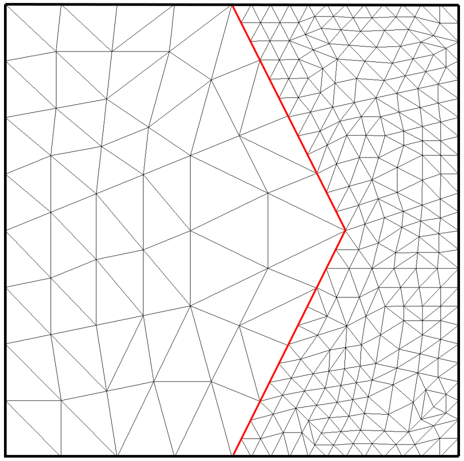
$$\ddot{\boldsymbol{u}}_i - \nabla \cdot \boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \boldsymbol{f}_i$$

$$\boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \lambda_i(\nabla \cdot \boldsymbol{u}_i)I + 2\mu_i\boldsymbol{\varepsilon}_i(\boldsymbol{u}_i)$$

$$\lambda = 0.864198 \quad \text{and} \quad \mu = 0.37037$$

Manufactured Solution

$$\boldsymbol{u}(\boldsymbol{x}, t) = (3\sin(x)\sin(y)\cos(t), \sin(x)\sin(y)t)$$



$L^2(\Omega)$ Error

$h_{min}(\Omega_1)$	$h_{min}(\Omega_2)$	Δt	IVR(1)	IVR(2)	IVR(12)
0.378545	0.113981	0.00371833	0.0146414	0.0146403	0.0146404
0.220723	0.0672413	0.00185917	0.00353829	0.00349268	0.00349301
0.107240	0.0359195	0.00101409	0.00095948	0.000854641	0.000854613
0.0514682	0.0196624	0.00053119	0.00033852	0.000217698	0.000217665
0.0277461	0.00957506	0.00024789	0.000141964	5.53096e-05	5.52471e-05
Rate			1.73	2.08	2.08

$H^1(\Omega)$ Error

$h_{min}(\Omega_1)$	$h_{min}(\Omega_2)$	Δt	IVR(1)	IVR(2)	IVR(12)
0.378545	0.113981	0.00371833	0.341327	0.340643	0.340643
0.220723	0.0672413	0.00185917	0.16736	0.16385	0.163848
0.107240	0.0359195	0.00101409	0.094672	0.081204	0.0812045
0.0514682	0.0196624	0.00053119	0.0701869	0.0404745	0.0404726
0.0277461	0.00957506	0.00024789	0.0576898	0.0204939	0.0204888
Rate			0.657	1.05	1.05

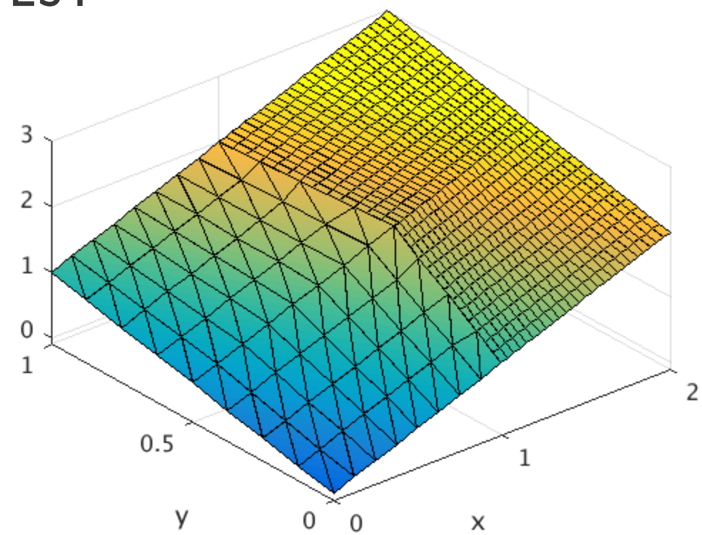
TRANSMISSION PROBLEM PATCH TEST

Advection Diffusion

$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i - \mathbf{u}\varphi_i) = f_i$$

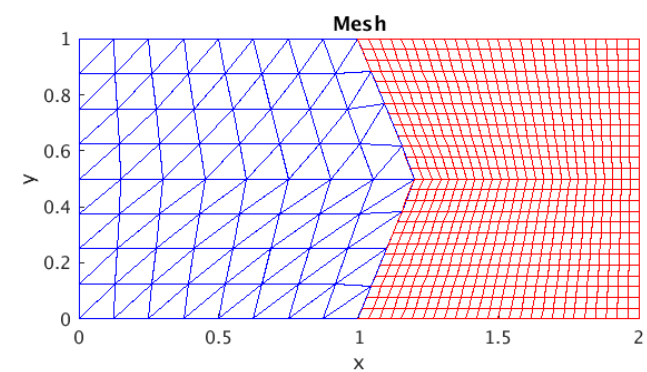
Manufactured Solution

$$\varphi_i(\mathbf{x}, t) = x + y$$



Pure Diffusion $\epsilon = 0.1$ $\mathbf{u} = 0$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	9.745e-04	1.062e-06	1.384e-13
$H^1(\Omega)$	4.089e-02	2.155e-05	3.106e-12



Moderate Advection $\epsilon = 0.1$ $\mathbf{u} = (-\sin(\pi/6), \cos(\pi/6))$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	8.417e-03	8.477e-06	2.229e-13
$H^1(\Omega)$	3.540e-01	1.523e-04	3.176e-12

Strong Advection $\epsilon = 0.0001$ $\mathbf{u} = (-\sin(\pi/6), \cos(\pi/6))$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	2.000e-01	3.175e-05	2.227e-13
$H^1(\Omega)$	1.136e+01	7.719e-04	4.341e-12

TRANSMISSION PROBLEM DISCONTINUOUS PATCH TEST

Advection Diffusion

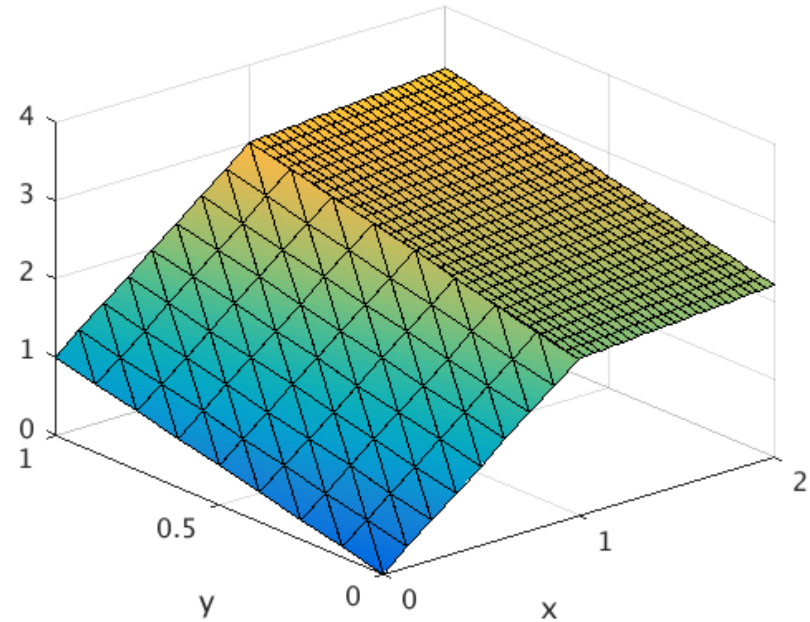
$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i - \mathbf{u}\varphi_i) = f_i$$

$$\epsilon_1 = 0.01 \qquad \epsilon_2 = 0.1$$

Manufactured Solution

$$\varphi_1(\mathbf{x}, t) = 2x + y$$

$$\varphi_2(\mathbf{x}, t) = 0.2x + y + 1.8$$

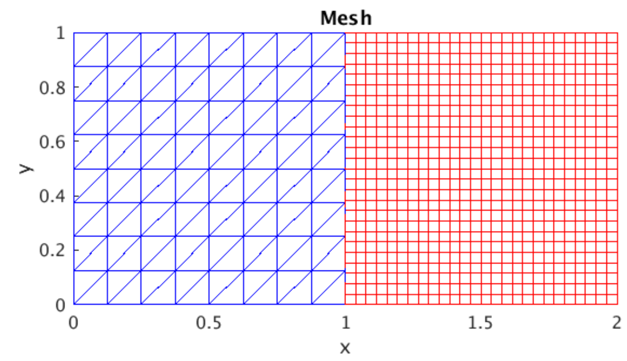


Pure Diffusion $\mathbf{u}_1 = \mathbf{u}_2 = 0$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	1.899e-04	3.963e-07	4.365e-14
$H^1(\Omega)$	7.510e-03	8.674e-06	1.920e-12

Moderate Advection $\mathbf{u}_1 = \mathbf{u}_2 = (-\sin(\pi/6), \cos(\pi/6))$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	1.269e-02	2.003e-05	1.700e-13
$H^1(\Omega)$	5.098e-01	3.573e-04	5.149e-12



TRANSMISSION PROBLEM CONVERGENCE

Advection Diffusion strong advection regime

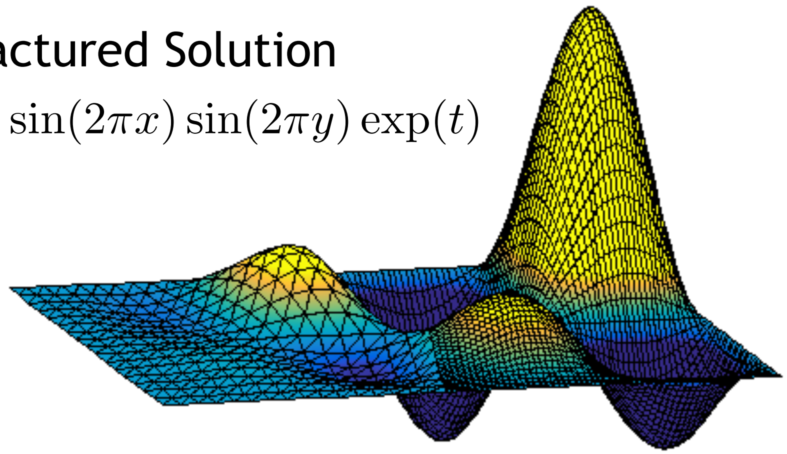
$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i - \mathbf{u}\varphi_i) = f_i$$

$$\mathbf{u} = (-\sin(\pi/6), \cos(\pi/6))$$

$$\epsilon = 0.0001$$

Manufactured Solution

$$\varphi_k(\mathbf{x}, t) = x^2 y \sin(2\pi x) \sin(2\pi y) \exp(t)$$



$L^2(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	1.042e-02	1.121e-01	9.114e-02	9.114e-02
0.125	0.0357	5.102e-03	3.482e-02	3.426e-02	3.783e-02
0.0625	0.0179	2.551e-03	8.282e-03	8.279e-03	8.747e-03
0.03125	0.00893	1.272e-03	1.620e-03	1.613e-03	1.663e-03
Rate	-	-	2.04	1.95	1.94

$H^1(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	1.042e-02	3.215	2.321	2.321
0.125	0.0357	5.102e-03	1.412	1.297	1.448
0.0625	0.0179	2.551e-03	0.6381	0.6261	0.6502
0.03125	0.00893	1.272e-03	0.3012	0.2985	0.3020
Rate	-	-	1.14	0.993	0.998

TRANSMISSION PROBLEM CONVERGENCE

Advection Diffusion
moderate advection regime

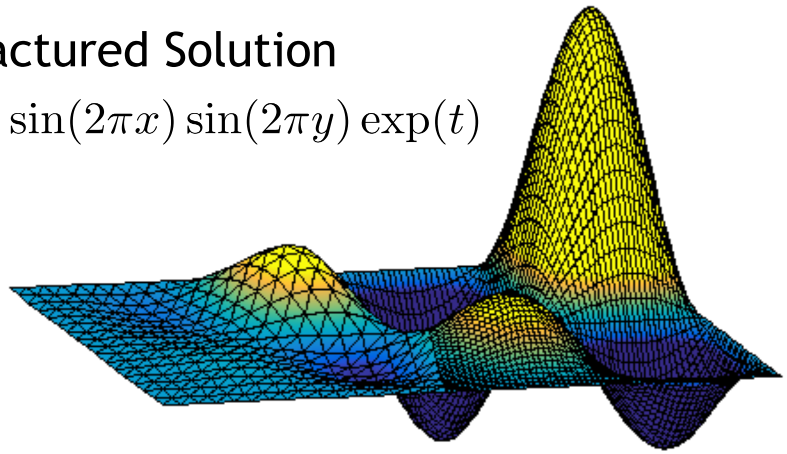
$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i - \mathbf{u}\varphi_i) = f_i$$

$$\mathbf{u} = (-\sin(\pi/6), \cos(\pi/6))$$

$$\epsilon = 0.1$$

Manufactured Solution

$$\varphi_k(\mathbf{x}, t) = x^2 y \sin(2\pi x)\sin(2\pi y)\exp(t)$$



$L^2(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	3.497e-03	9.021e-02	9.043e-02	9.043e-02
0.125	0.0357	8.681e-04	2.876e-02	2.891e-02	3.367e-02
0.0625	0.0179	2.161e-04	8.157e-03	8.181e-03	8.655e-03
0.03125	0.00893	5.391e-05	2.133e-03	2.136e-03	2.186e-03
Rate	-	-	1.80	1.80	1.81

$H^1(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	3.497e-03	2.080	2.072	2.072
0.125	0.0357	8.681e-04	1.096	1.094	1.124
0.0625	0.0179	2.161e-04	0.5602	0.5594	0.5652
0.03125	0.00893	5.391e-05	0.2821	0.2817	0.2831
Rate	-	-	0.962	0.960	0.961

TRANSMISSION PROBLEM CONVERGENCE

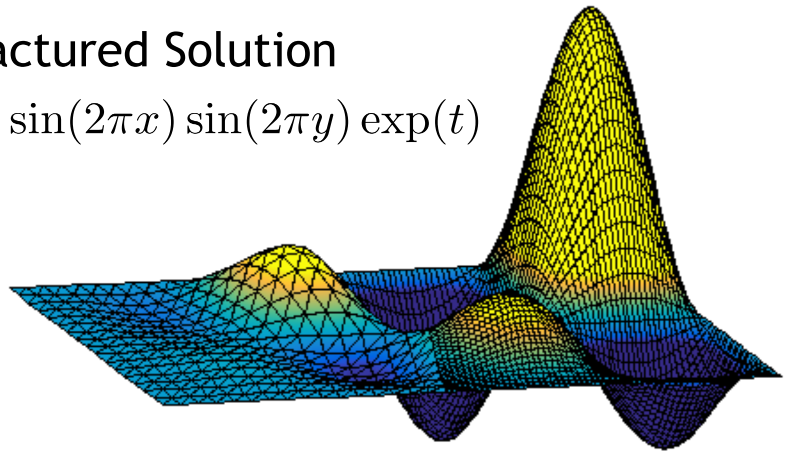
Pure Diffusion

$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i) = f_i$$

$$\epsilon = 0.1$$

Manufactured Solution

$$\varphi_k(\mathbf{x}, t) = x^2 y \sin(2\pi x)\sin(2\pi y)\exp(t)$$



$L^2(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	3.497e-03	9.573e-02	9.630e-02	9.630e-02
0.125	0.0357	8.681e-04	3.049e-02	3.061e-02	3.531e-02
0.0625	0.0179	2.161e-04	8.636e-03	8.660e-03	9.180e-03
0.03125	0.00893	5.391e-05	2.257e-03	2.261e-03	2.320e-03
Rate	-	-	1.80	1.81	1.81

$H^1(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	3.497e-03	2.049	2.049	2.049
0.125	0.0357	8.681e-04	1.092	1.091	1.116
0.0625	0.0179	2.161e-04	0.5594	0.5587	0.5643
0.03125	0.00893	5.391e-05	0.2820	0.2816	0.2830
Rate	-	-	0.955	0.956	0.955

CONCLUSIONS

Partitioned Implicit Value Recovery (IVR) Scheme

- Uses a well-posed monolithic mixed formulation to estimate boundary data
- Key idea is to consider alternative constraint, which enables explicit treatment of Lagrange multiplier
- Results in non-iterative partitioned method
- Stability and accuracy derive from the stability and accuracy of the mixed method
- Subdomain finite element spaces on the interface are stable choices for the Lagrange multiplier

Next steps

- Extend to non-coincident interfaces
- Investigate alternative coupling conditions

$$M_1 \ddot{\mathbf{u}}_1 + G_1^T \mathbf{t} = \mathbf{f}_1(\mathbf{u}_1)$$

$$M_2 \ddot{\mathbf{u}}_2 - G_2^T \mathbf{t} = \mathbf{f}_2(\mathbf{u}_2)$$

$$G_1 \ddot{\mathbf{u}}_1 - G_2 \ddot{\mathbf{u}}_2 = 0$$

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