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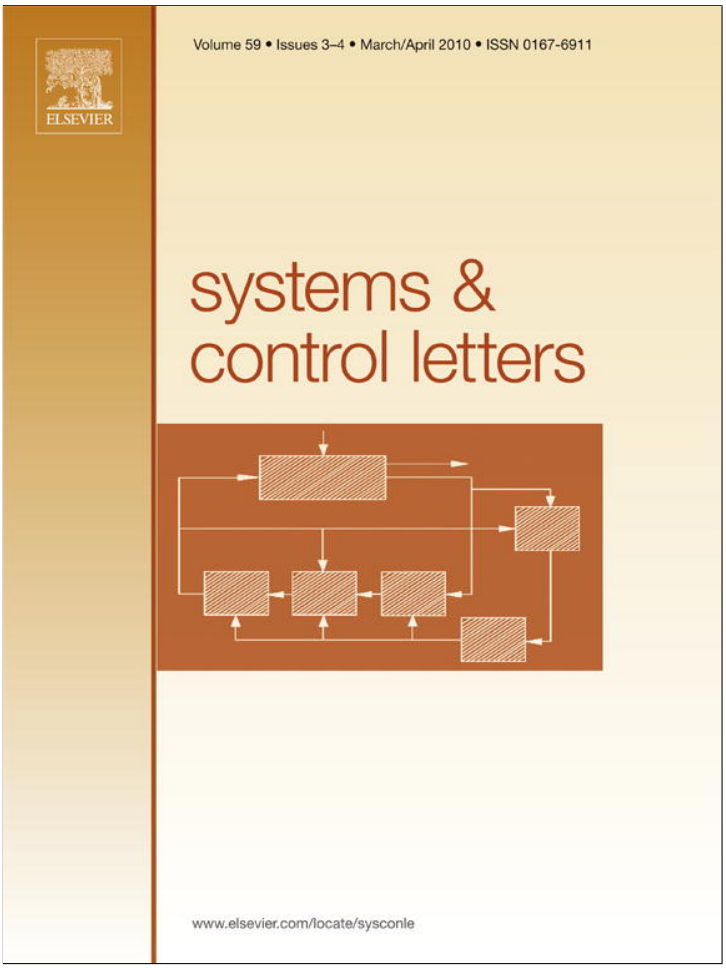

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Optimal controller synthesis for a class of LTI systems via switched feedback

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ABSTRACT

We develop a switched feedback controller that optimizes the rate of convergence of the state trajectories to the origin for a class of second order LTI systems. Specifically, we derive an algorithm which optimizes the rate of convergence by employing a controller that switches between symmetric gains. As a byproduct of our investigation, we find that, in general, the controllers which optimize the rate of convergence switch between two linear subsystems, one of which is unstable. The algorithm we investigate will design optimal switching laws for the specific case of second order LTI plants of relative degree two.

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1. Introduction

Stabilization of continuous time systems via both switched and hybrid feedback is a problem that has received much attention in the recent literature (see, e.g. [1–16]). In particular, much attention has been given to the specific problem in which a second order linear plant (which is assumed not to be static output feedback stabilizable) can be stabilized by employing a control law that switches between *multiple* static output feedback laws. Hu et al. gave a partial answer to this question for second order systems in [4] based upon the conic switching laws of [16]. Benassi et al. in [2] and Litysn et al. in [9] derived necessary and sufficient conditions for a second order linear plant to be stabilized via a hybrid feedback automaton based upon a specific eigenvalue criterion. In our prior work [11], we provide necessary and sufficient conditions on the stabilizability of second order systems via a particular nonlinear output feedback law. The main result, repeated here, is as follows:

Theorem 1.1. *Consider the system*

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1.1}$$

with $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$, and $C \in \mathbb{R}^{1\times 2}$, where neither C nor B is identically 0. Define the root locus of this system to be the locus of eigenvalues of $A + kBC$ as k varies continuously over $\mathbb{R}$. Then exactly one of the following statements is true:

1. *The system is static output feedback stabilizable.*
2. *The system is not static output feedback stabilizable, but it has root locus which takes on complex values for some values of $k \in \mathbb{R}$ and is stabilizable by a control law of the form $u(x) = v(x)Cx$ with*

$$v(x_1, x_2) = \begin{cases} k_1 & w'x = 0 \\ k_2 & w'x \neq 0. \end{cases}$$

   Here, $w'q = 0$, where q is the sole stable, real eigenvector of the matrix $A + k_1BC$, and where k_2 is chosen such that the eigenvalues of $A + k_2BC$ are complex.
3. *The system has a root locus which is real for all values of $k \in \mathbb{R}$ and is not stabilizable by control of the form $u(x) = v(x)Cx$ for any choice of $v(x)$.*

When it is possible, Theorem 1.1 provides a constructive method of designing a stabilizing controller which implements either static or switched output feedback. Such a result is attractive since switching between static gains is a simple form of control; however, the result is not discriminatory in the sense that, if a given second order system satisfies either the first or second item in Theorem 1.1, there are *several* controllers which achieve stability. It is therefore natural to ask the question as to whether there is a smaller class of controllers which are easily characterizable and which can be considered "good" in some sense.

The problem which we consider in this paper is the following: we consider a family of state-dependent controllers of the form $u(x) = v(x)Cx$ where the scalar function $v(x)$ is homogeneous of degree zero and lies in a bounded, symmetric interval (i.e., $v(x) \in [-v_0, v_0]$ for some $v_0 > 0$). Using this family of controllers, we then explicitly characterize a set of switching controllers which maximize the *rate of convergence* (to be defined formally in the next section) of the state trajectory to the origin.

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Note that the problem of studying convergence rates is not a new one as, in [10], it is shown that the Lyapunov exponent for a class of second order systems in feedback with a particular linear hybrid feedback controller can be made arbitrarily large by making a particular control parameter as large as is necessary. Independent of the difference of technique/control architecture that we use here vs. what is used in [10], the result we present here can be viewed as an extension of this prior work in that sense that, when the controller gain is *bounded*, we provide a controller which achieves the largest rate of convergence subject to the given control bound.

After we formally define the optimization problem to be considered, we shall begin by finding an optimal controller for a particular state–space description of the plant $P(s)$. We shall then show that controller designs in arbitrary state–space descriptions can be extracted via an appropriate change of coordinates, and an explicit algorithm for designing controllers in arbitrary state–space descriptions will be presented. Due to space constraints, several proofs and discussions have been curtailed or omitted (the reader is referred to Chapter 3 of [15] for a more detailed treatment of the subject matter presented here).

An interesting qualitative byproduct of the work we perform here is the following: whenever the bound on the switching gain v_0 is sufficiently large, the controllers which optimize the rate of convergence end up switching between two linear subsystems, one of which is exponentially *unstable*. This result may at first appear counterintuitive, but as we show here, the optimal controllers that we propose operate by driving the state of the system onto a stable manifold of the unstable subsystem, similar to the manner in which a sliding mode controller operates.

During the preparation of this manuscript, it was pointed out to us that Theorem 1.1 is very similar to the results of both [2,9], with a small difference in that we employ a framework that uses continuous-time nonlinear feedback laws while [2,9] examine controllers which employ hybrid feedback automata. Hence, while our initial result 1.1 was derived without knowledge of either of these prior results, both [2,9] should be viewed as original sources of the work upon which we expand in this document.

2. Problem formulation

2.1. Rate of convergence: Definition

Recall that a system described by $\dot{x} = f(x)$ is said to be *globally exponentially stable* if there exist constants $M, \beta > 0$ such that, for all solutions $x(t)$,

$$\|x(t)\| \le M e^{-\beta t}\|x(0)\| \quad \forall t \ge 0. \tag{2.2}$$

For a globally exponentially stable autonomous system of the form $\dot{x} = f(x), x(0) = x_0$ where $f(x)$ is homogeneous and piecewise continuous, we define the *rate of convergence* R as

$$R = \min_{\|x_0\|=1} \liminf_{T\to\infty} -\frac{1}{2T}\ln\left(\|x(T)\|^2\right).$$

2.2. Problem: Maximum rate of convergence

Consider a second order single-input, single-output LTI system of relative degree two of the form

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{2.3}$$

and further consider a feedback control law of the form $u(x) = v(x)Cx$, where $u(x)$ is homogeneous, so that the overall interconnected system is an autonomous system which takes the form

$$\dot{x} = Ax + v(x)BCx, \qquad x(0) = x_0\text{: given.} \tag{2.4}$$

Here, the scalar function $v(x)$ satisfies the condition $v(x) \in [-v_0, v_0]$ $\forall x \in \mathbb{R}^2$ with $v_0 > 0$ such that:

- $\exists v_1 \in [-v_0, v_0]$ such that the eigenvalues of $A + v_1BC$ form a complex conjugate pair.
- $\exists v_2 \in [-v_0, v_0]$ such that at least one of the eigenvalues of $A + v_2BC$ lies strictly in the open left half plane.

It is easily verified that any choice of $v(x)$ that satisfies the bulleted criteria above admits a stabilizing controller as described in [11].

It is clear that, for each choice of $v(x)$, the autonomous system Eq. (2.4) has an associated rate of convergence R. For a given plant Eq. (2.3) and given value v_0, the task at hand is to find a choice of $v(x)$ such that the corresponding rate R is maximal. We use the notation

$$R^*(v_0) = \max_{v(x)\in[-v_0,v_0]} R(v(x))$$

to denote this optimal value.

In the sections that follow, we find a choice of $v^*(x) \in [-v_0, v_0]$ which achieves the maximal rate in the above optimization problem and also explicitly characterize $R^*(v_0)$ in terms of v_0 and the parameters of the plant transfer function $C(sI - A)^{-1}B$. We prove optimality of the resulting controllers by first finding an optimal controller for a particular state–space realization of a plant with transfer function $P(s)$ and then use this to form a general design algorithm for arbitrary minimal state–space descriptions.

3. Optimal controller synthesis for plants of relative degree two

In this section, we consider the design of a controller which maximizes the rate of convergence for a second order plant of the form

$$P(s) = \frac{c}{s^2 + as + b} \tag{3.5}$$

where a, b, and $c \in \mathbb{R}$. Without loss of generality, we focus on the case where $c = 1$. We first consider the problem of control design for the case where the plant has state–space description given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} -\frac{a}{2} & \gamma \\ \frac{-b+\frac{a^2}{4}}{\gamma} & -\frac{a}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{\gamma}\end{bmatrix}u \tag{3.6}$$

$$y = x_1 \tag{3.7}$$

where $\gamma > 0$ is a free parameter that we shall choose. Under the feedback law $u = vx_1$, the characteristic polynomial of the closed-loop system has roots

$$s = \frac{-a \pm \sqrt{a^2 - 4b + 4v}}{2}.$$

A straightforward calculation shows that, in order for the system to be stabilizable via the switching algorithm of [11], v_0 must satisfy the following condition:

$$v_0 > \begin{cases} \left|\frac{a^2}{4} - b\right|, & a > 0 \\ \max\left\{\frac{a^2}{4} - b, b\right\}, & a \le 0. \end{cases} \tag{3.8}$$

3.0.1. Upper bound on optimal rate of convergence

Our first goal is to prove the following statement:

Theorem 3.2. *Consider the system of Eqs. (3.6) and (3.7) under the feedback law $u(x) = v(x)y$ with $v(x) \in [-v_0, v_0]$ for all $x \in \mathbb{R}^2$ where v_0 satisfies the conditions of Eq. (3.8). Suppose that v_0 satisfies the additional constraint that*

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$$v_0 \geq 2b - \frac{a^2}{2}. \tag{3.9}$$

Then the optimal rate of convergence $R^*(v_0)$ satisfies the inequality

$$R^*(v_0) \leq -\lambda_{\min}(A + v_0 BC)$$

where A, B, and C are the corresponding matrices of the state–space description in Eqs. (3.6) and (3.7), and $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a square matrix.

In the sequel we show that the above upper bound can be achieved with equality by an appropriate controller selection.

Proof. Let $\tilde{A}$ be given by

$$\tilde{A}(v(x)) = A + v(x)BC = \begin{bmatrix} -\frac{a}{2} & \gamma \\ \frac{-b + \frac{a^2}{4} + v(x)}{\gamma} & -\frac{a}{2} \end{bmatrix}.$$

Along the trajectories of the system we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 &= x(t)'\left(\tilde{A}(v(x(t))) + \tilde{A}'(v(x(t)))\right)x(t) \\ &\geq \min_{|v|\leq v_0} \lambda_{\min}\left(\tilde{A}(v) + \tilde{A}'(v)\right)\|x(t)\|^2. \end{aligned}$$

Now, a simple calculation shows that

$$\lambda_{\min}\left(\tilde{A}(v) + \tilde{A}'(v)\right) = -a - \left|\gamma + \frac{v - b + \frac{a^2}{4}}{\gamma}\right|.$$

Note that the lower bound we derived on the derivative of the squared Euclidean norm of $x(t)$ must hold for *any* $\gamma > 0$. Hence, we can fix γ to be a convenient value and compute the corresponding minimum in the above expression. If we choose $\gamma = \sqrt{\frac{a^2}{4} - b + v_0}$, we find that the minimum of the above expression occurs at $v = v_0$ and is given by

$$\min_{|v|\leq v_0} \lambda_{\min}\left(\tilde{A}(v) + \tilde{A}'(v)\right) = 2\lambda_{\min}(A + v_0 BC) \triangleq 2\tilde{\lambda}.^{1}$$

Solving the resulting differential inequality, we find $\|x(t)\|^2 \geq e^{2\tilde{\lambda}t}\|x(0)\|^2$. Now,

$$\begin{aligned} R^*(v_0) &= \min_{\|x(0)\|=1} \liminf_{T\to\infty} -\frac{1}{2T}\ln\left(\|x(T)\|^2\right) \\ &\leq \min_{\|x(0)\|=1} \liminf_{T\to\infty} -\frac{1}{2T}\ln\left(e^{2\tilde{\lambda}T}\|x(0)\|^2\right) \\ &= -\tilde{\lambda}. \quad \square \end{aligned}$$

3.0.2. Achieving the upper bound: Optimal controller structure

We now derive a generic control law $v(x)$ for which the corresponding rate of convergence achieves the upper bound of Theorem 3.2. We first find a controller $v(x)$ for the state–space description of Eqs. (3.6) and (3.7), and then show how to derive optimal controllers for arbitrary state–space descriptions in a later section.

As it turns out, a controller which achieves the upper bound on the rate of convergence we derived in the previous section is a so-called "bang–bang" controller, i.e., a controller which switches between the two extreme values, v_0 and $-v_0$. To begin, we first find the eigenspace of the matrix $A + v_0 BC$ where A, B, and C are the state–space matrices of Eqs. (3.6) and (3.7) when $\gamma = \sqrt{\frac{a^2}{4} - b + v_0}$:

$$A + v_0 BC = \begin{bmatrix} -\frac{a}{2} & \gamma \\ \gamma & -\frac{a}{2} \end{bmatrix},$$

which has eigenvalues $\lambda(A + v_0 BC) = -\frac{a}{2} \pm \gamma$. If we denote the minimum and maximum eigenvalues as λ_s and λ_u, respectively, then the corresponding eigenvectors w_s and w_u are given by

$$w_s = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad w_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.10}$$

The basic algorithm that we use to achieve stability (and the upper bound on the rate of convergence) is essentially the same as the one used in [11]: we choose a value of gain that yields complex eigenvalues to induce rotation and switch the value of the gain to v_0 when the state trajectory lands on the stable manifold. The exact algorithm we use is described in the following theorem:

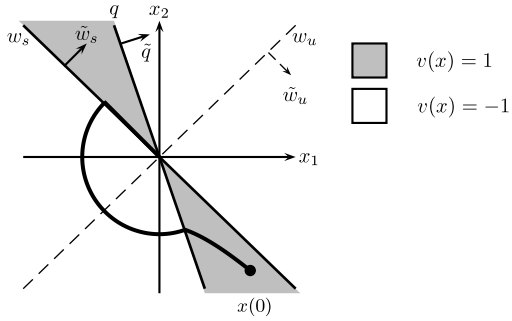

Fig. 3.1. Graphical depiction of switching law of Eq. (3.11).

Theorem 3.3. *Let w_s and w_u be given as in Eq. (3.10), and let $\tilde{w}_s$ and $\tilde{w}_u$ be defined such that $\tilde{w}_s' w_s = 0$ and $\tilde{w}_u' w_u = 0$ where $\tilde{w}_s$ and $\tilde{w}_u$ are both oriented in a clockwise orientation. Suppose that $q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}'$ satisfies the conditions $q_2 > 0$ and*

$$(\tilde{w}_s' q)(\tilde{w}_u' q) < 0,$$

and let $\tilde{q}$ be defined in such a way that $\tilde{q}' q = 0$ where $\tilde{q}$ is oriented clockwise. Then the control law

$$v(x) = \begin{cases} v_0 & x'(\tilde{w}_s \tilde{q}')x \leq 0 \\ -v_0 & x'(\tilde{w}_s \tilde{q}')x > 0 \end{cases} \tag{3.11}$$

asymptotically stabilizes the plant of Eqs. (3.6) and (3.7) when $\gamma = \sqrt{a^2/4 - b + v_0}$ with rate

$$R = -\lambda_{\min}(A + v_0 BC).$$

An immediate corollary to Theorem 3.3 is the following.

Corollary 1. *The optimal rate of convergence for the plant of Eqs. (3.6) and (3.7) when $v(x)$ is constrained to lie in the range $[-v_0, v_0]$ is*

$$R^*(v_0) = -\lambda_{\min}(A + v_0 BC).$$

The control law of Eq. (3.11) is depicted graphically in Fig. 3.1. Essentially, if we pick any vector q that lies to the right of one of the eigenvectors and to the left of the other, then this induces a partition on the state–space where we use the gain $v(x) = v_0$ in the region bound by the w_s and q, and where we use $v(x) = -v_0$ in the region bound by w_u and q. A sample phase portrait for an

[1] For the selected value of γ, note that $A + v_0 BC$ is symmetric and, hence, has real eigenvalues.

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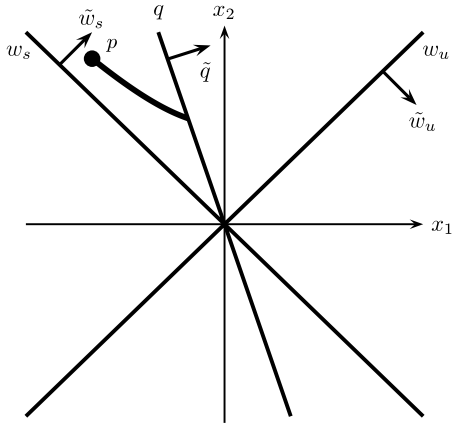

Fig. 3.2. Relative positioning of w_s, w_u, q, and p for Lemma 3.1.

initial condition $x(0)$ which lies in the region of the state–space where $v(x) = v_0$ is depicted in the figure, as well.

In order to prove Theorem 3.3, we need the result of the following lemma.

Lemma 3.1. *Consider a diagonalizable matrix $A \in \mathbb{R}^{2\times 2}$ which has real eigenvalues λ_s and λ_u, $\lambda_s < \lambda_u$, and corresponding eigenvectors w_s and w_u, respectively. Define $\tilde{w}_s$ and $\tilde{w}_u$ in such a way that $\tilde{w}_s' w_s = 0$ and $\tilde{w}_u' w_u = 0$, and consider vectors q and p that satisfy the following conditions: $\tilde{w}_s' q > 0$, $\tilde{w}_u' q < 0$, $\tilde{w}_s' p > 0$, $\tilde{q}' p < 0$, where $\tilde{q}$ satisfies $\tilde{q}' q = 0$. Assume that $\tilde{w}_s$, $\tilde{w}_u$, and $\tilde{q}$ are all oriented in the same direction (either clockwise or counterclockwise) and that $\tilde{w}_s' w_u \geq 0$. Then there exists $t > 0$ such that the state trajectory $x(t)$ of the system*

$$\dot{x} = Ax, \qquad x(0) = p$$

satisfies the condition $x(t) = \alpha q$ for some $\alpha \in \mathbb{R}$.

A geometric interpretation of Lemma 3.1 is shown in Fig. 3.2. The conditions in the lemma reflect the relative positioning of q and p with respect to the two eigenvectors w_s and w_u. The conditions on p and q ensure that there is no eigenvector between p and q, and that p is closer in angle to w_s than q. Since $\lambda_u > \lambda_s$, one should expect the phase portraits to approach w_u as $t \to \infty$. The overall statement, then, reflects the following intuitive notion: since p lies closer to the eigenvector with smaller eigenvalue λ_s than the vector q, the state trajectory $x(t)$ must cross the line defined by the vector q at some finite, positive time. The proof of this statement can be found in [11].

Proof of Theorem 3.3. We shall first show the following: the closed loop system $\dot{x} = (A + v(x)BC)x$ satisfies the condition that, for every initial condition $x(0)$, there exists $t_0 \geq 0$ such that $x(t_0) = \alpha w_s$ for some $\alpha \in \mathbb{R}$, i.e., that every initial condition is driven onto the stable manifold in finite time. We first consider the case when $x(0)$ lies in the region

$$x(0)'(\tilde{w}_s\tilde{q}')x(0) > 0.$$

In this region, we have the $v(x(0)) = -v_0$. The eigenvalues of the matrix $A - v_0BC$ in this region are of the form $-a/2 \pm j\omega_0$ where $\omega_0 = \sqrt{v_0 + b - a^2/4}$. Now, it is clear that there exists some time $t_0 < \pi/\omega_0$ for which $x(t_0)'(\tilde{w}_s\tilde{q}')x(t_0) = 0$ and for which $v(x(t)) = -v_0$ for all $t < t_0$. This, in turn, implies that one of the following conditions holds

$$\tilde{w}_s' x(t_0) = 0 \quad \text{or} \quad \tilde{q}' x(t_0) = 0.$$

We shall show that the former condition must hold by showing that the latter condition is impossible. Suppose that at time t_0, $\tilde{q}' x(t_0) = 0$. We may equivalently write this condition as $x(t_0) = \beta \begin{bmatrix} q_1 & q_2 \end{bmatrix}'$, where $\beta \in \mathbb{R}$, and let $\tilde{q}$ be the clockwise oriented vector $\tilde{q} = \begin{bmatrix} q_2 & -q_1 \end{bmatrix}'$. Now,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)'(\tilde{w}_s\tilde{q}')x(t)\right) = x(t)'(A - v_0BC)'(\tilde{w}_s\tilde{q}')x(t) + x(t)'(\tilde{w}_s\tilde{q}')(A - v_0BC)x(t). \tag{3.12}$$

If Eq. (3.12) is satisfied, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)'(\tilde{w}_s\tilde{q}')x(t)\right)\bigg|_{t=t_0} = \beta^2(q_1 + q_2)\left(\frac{\omega_0^2}{\gamma}q_1^2 + \gamma q_2^2\right).$$

Recalling the conditions $(\tilde{w}_s' q)(\tilde{w}_u' q) < 0$ and $q_2 > 0$, we find that $q_1 + q_2 > 0$ which means that the above derivative is positive. But if $\tilde{q}' x(t_0) = 0$ and the above derivative is positive, then it follows that $x(t_0^-)(\tilde{w}_s\tilde{q}')x(t_0^-) < 0$ which implies that $v(x(t_0^-)) = v_0$, i.e., that a switch from $-v_0$ to v_0 has already occurred, which is an obvious contradiction. Hence, we conclude that there exists $t_0 < \pi/\omega_0$ such that $\tilde{w}_s' x(t_0) = 0$ or, equivalently, $x(t_0) = \alpha w_s$ for some $\alpha \in \mathbb{R}$.

Now consider the case where the initial condition $x(0)$ satisfies $x(0)'(\tilde{w}_s\tilde{q}')x(0) \leq 0$. We shall break this condition down into three separate cases

1. $x(0) = \alpha w_s, \alpha \in \mathbb{R}$.
2. $x(0) = \beta q, \beta \in \mathbb{R}$.
3. $x(0)'(\tilde{w}_s\tilde{q}')x(0) < 0$.

The first case immediately yields the result that we desire. For the second case, the analysis above shows that if $x(0) = \beta q$ for some $\beta \in \mathbb{R}$, then $x(0^+)'(\tilde{w}_s\tilde{q}')x(0^+) > 0$. Employing the time-invariance of the interconnected system, the problem now reduces to showing that there exists $t_0 > 0$ such that the state trajectory $\hat{x}(t)$ with initial state $\hat{x}(0) = x(0^+)$ satisfies $\hat{x}(t) = \alpha w_s$ for some $\alpha \in \mathbb{R}$, which we already showed above.

To consider the final case, we shall use the result of Lemma 3.1. By our assumptions, we have that $\tilde{w}_s$, $\tilde{w}_u$, and $\tilde{q}$ are all oriented in the same direction (clockwise), and we also have that $\tilde{w}_s' q > 0$, $\tilde{w}_u' q < 0$. Now, if we let $p = x(0)$, one of the following pairs of conditions must hold: either $\tilde{w}_s' p > 0$ and $\tilde{q}' p < 0$, or $\tilde{w}_s' p < 0$ and $\tilde{q}' p > 0$. Because the interconnected system is homogeneous, we may assume without loss of generality that the first set of conditions holds.

Under these assumptions, Lemma 3.1 guarantees that there exists some time $t_1 > 0$ for which $x(t) = \beta q$ for some $\beta \in \mathbb{R}$. But if we now consider the new initial condition $z(0) = x(t_1)$, we know from the second case that there exists t_2 such that $z(t_2) = x(t_1 + t_2) = \alpha w_s, \alpha \in \mathbb{R}$. Hence, we may take $t_0 = t_1 + t_2$.

Now, if we let $\tilde{\lambda} = \lambda_{\min}(A + v_0BC) < 0$, we see that, for every initial condition $x(0)$, there exists $t_0 \geq 0$ and $\alpha \in \mathbb{R}$ such that $x(t) = \alpha e^{\tilde{\lambda}(t-t_0)} w_s$ from which asymptotic stability immediately follows. To establish the result on the corresponding rate R, we note that

$$\liminf_{T\to\infty} -\frac{1}{2T}\ln\left(\|x(T)\|^2\right) = \lim_{T\to\infty} -\frac{1}{2T}\ln\left(e^{2\tilde{\lambda}(T-t_0)}\|w_s\|^2\right) = -\tilde{\lambda}.$$

Since this results holds true for all $x(0) \in \mathbb{R}^2$, we conclude that

$$R = -\lambda_{\min}(A + v_0BC). \quad \square$$

A few comments are in order. First, it is clear that controllers that achieve the optimal rate are not unique; the parameter q is a free design parameter (subject to the constraints imposed in Theorem 3.3). While we shall not discuss this in a formal manner, it is typically that case that one generally chooses q to be sufficiently far (in an angular sense) from both of the eigenvectors w_s and w_u. Choosing q to be very close to w_u leads to "practical" instability since, for initial conditions that lie in the shaded area of Fig. 3.1,

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the Euclidean norm of the state vector may grow very large before the gain is switched from v_0 to $-v_0$. Choosing q too close to w_s, on the other hand, can lead to robustness issues if *time delays* are present in the systems under consideration (see Chapter 4 of [15] for an explanation of this particular matter).

Remark 3.1. Note that when $v_0 > b$ in addition to the assumptions of Eqs. (3.8) and (3.9), one eigenvalue of the matrix $A + v_0BC$ is real and positive. Hence, for v_0 sufficiently large, the optimal controllers of Eq. (3.11) (which achieve the "most" stable closed-loop system in terms of convergence rate) switch between two linear subsystems, one of which is exponentially *unstable*.

The fact that *every* initial condition can be driven onto the stable manifold in finite time using the control law of Eq. (3.11) yields the following corollary whose proof is immediate:

Corollary 2. *The system of Eqs. (3.6) and (3.7) under the control action of Eq. (3.11) satisfies the condition*

$$\liminf_{T\to\infty} -\frac{1}{2T}\ln\left(\|x(T)\|^2\right) = R^*(v_0)$$

for every *initial condition* $x(0) \neq 0$.

One of the significant results of Corollary 2 is that it implies our definition of rate of convergence (as supplied in Section 2) is equivalent to the well-studied notion of the *Lyapunov exponent* (see, e.g., [5]) when considering systems of the form Eqs. (3.6) and (3.7) under the control action of Eq. (3.11). There is a minor subtlety in that the Lyapunov exponent is defined for individual trajectories (i.e., it is defined as a function of the initial condition $x(0)$ and can, hence, possibly take on different values for different initial conditions), but Corollary 2 establishes that the Lyapunov exponent is uniform over all initial conditions for the specific class of systems being considered, and, hence, the two are equivalent notions in this case.

3.0.3. Optimal controllers for arbitrary state–space descriptions

To obtain an optimal design for all other state–space realizations of a given second order transfer function of relative degree two, essentially, one need only apply a simple change of coordinates:

Proposition 3.1. *Consider an exponentially stable system of the form* $\dot{x} = Ax + v(x)BCx$ *with rate of convergence* R *where* $v(x)$ *takes the form*

$$v(x) = \begin{cases} v_1 & x'F_1F_2'x \le 0 \\ v_2 & x'F_1F_2'x > 0 \end{cases}$$

where F_1, F_2 *are column vectors of appropriate dimension. Then the system* $\dot{z} = \tilde{A}z + \tilde{v}(z)\tilde{B}\tilde{C}z$ *with* $\tilde{v}(z)$ *given by*

$$\tilde{v}(z) = \begin{cases} v_1 & z'\tilde{F}_1\tilde{F}_2'z \le 0 \\ v_2 & z'\tilde{F}_1\tilde{F}_2'z > 0 \end{cases}$$

with

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B, \qquad \tilde{C} = CT,$$
$$\tilde{F}_i = T'F_i, \quad i = 1, 2$$

where T *is an invertible matrix is also exponentially stable with convergence rate* R.

Proof. Performing the change of coordinates $z = Mx$ yields the above expressions for the matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{F}_1$, and $\tilde{F}_2$. What remains to be shown is that the rate of convergence is invariant to a coordinate change. In the new system of coordinates, the rate of convergence R' is given by

$$R' = \min_{\|z(0)\|=1} \liminf_{T\to\infty} -\frac{1}{2T}\ln\left(\|z(T)\|^2\right)$$
$$= \min_{\|Mx(0)\|=1} \liminf_{T\to\infty} -\frac{1}{2T}\ln\left(x(T)'M'Mx(T)\right).$$

Now, for every initial condition $x(0)$ and every $T > 0$, we have

$$-\frac{1}{2T}\ln\left(\lambda_{\max}(M'M)\right) + r(T) \le R' \le -\frac{1}{2T}\ln\left(\lambda_{\min}(M'M)\right) + r(T)$$

where $r(T)$ is given by

$$r(T) = -\frac{1}{2T}\ln\left(\|x(T)\|^2\right).$$

Applying the squeeze theorem to the above inequalities, we can conclude that

$$R' = \min_{\|Mx(0)\|=1} \liminf_{T\to\infty} -\frac{1}{2T}\ln\left(\|x(T)\|^2\right).$$

Now, by Corollary 2, we have that

$$R = \liminf_{T\to\infty} -\frac{1}{2T}\ln\left(\|x(T)\|^2\right)$$

for *every* initial condition $x(0)$, from which it immediately follows that $R' = R$. □

We shall now describe a method which does not require an explicit change of coordinates, i.e., a method of designing switching controllers directly in the coordinate frame of interest.

The control laws we have constructed take the form shown in Proposition 3.1 with $F_1 = \tilde{w}_s$ and $F_2 = \tilde{q}$. From the proof of Lemma 3.1, we see that an invertible change of coordinates will yield the following results, the proofs of which are immediate and left to the reader:

- If $F_1 = \tilde{w}_s$ is a vector which is normal to the stable eigenvector of $A + v_0BC$, then the transformed vector $\tilde{F}_1$ is a vector which is normal to the stable eigenvector of $T^{-1}(A + v_0BC)T$.
- If $\tilde{w}_s$, $\tilde{w}_u$ and q satisfy the relationships $\tilde{w}_s'q > 0$ and $\tilde{w}_u'q < 0$, then the transformed vectors $\hat{\tilde{w}}_s'\hat{q} > 0$ and $\hat{\tilde{w}}_q'\hat{q} < 0$ where $\hat{\tilde{w}}_s = T'\tilde{w}_s$, $\hat{\tilde{w}}_u = T'\tilde{w}_u$, and $\hat{q} = T^{-1}q$.

In layman's terms, the above conditions tell us that, for *any* state–space description, one can choose an optimal controller by choosing one switching boundary to be the stable eigenvector and choosing the other boundary to be a vector q that lies "between" the stable eigenvector and the unstable eigenvector. Specifically, we pick q to lie in the region for which the state trajectories for both $\dot{x} = (A + v_0BC)x$ and $\dot{x} = (A - v_0BC)x$ will move in the same direction across the switching boundary which, in practice, can be done via a graphical examination of the corresponding phase portraits.

Example 3.1. Consider the unstable LTI plant

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -12 & 7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$
$$y = x_1$$

and the associated task of finding a switching controller $v(x)$ which achieves maximal convergence rate for $|v(x)| \le 99.75$. We begin by computing the eigenvectors of the matrix $A + v_0BC$:

$$w_s = \begin{bmatrix}-1\\ 6.5\end{bmatrix}, \qquad w_u = \begin{bmatrix}1\\ 13.5\end{bmatrix}$$

which are depicted graphically in Fig. 3.3. In order to determine the region where we should place q (either the region bound by the eigenvectors which contains the x_2 axis, or the region bound by

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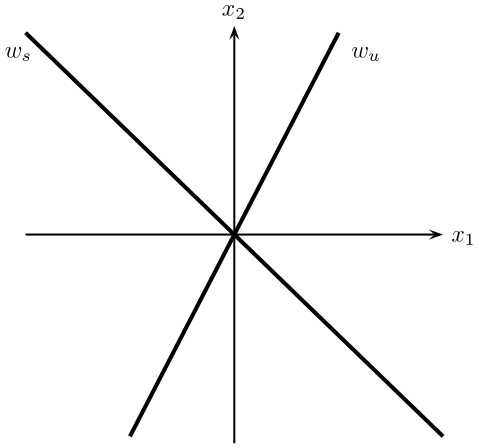

Fig. 3.3. Eigenvectors for the matrix $A + v_0BC$ for Example 3.1 (not drawn to scale).

the eigenvectors that contains the x_1 axis), we first determine the orientation of rotation for the phase portraits of $A - v_0BC$. Note that $\dot{x} = (A - v_0BC)x$ satisfies the property that, when $x_1 = 0, \dot{x_1} = x_2$. From this, we immediately conclude that the phase portraits of $\dot{x} = (A - v_0BC)x$ are rotating in a clockwise manner. By examining the relative placement of the eigenvectors in Fig. 3.3, we see that the phase portraits of $\dot{x} = (A + v_0BC)x$ are rotating in a clockwise manner in the region which contains the positive x_2 axis. Hence, we may choose any q in this region to define our control law. Picking $q = \begin{bmatrix}0 & 1\end{bmatrix}'$(and, correspondingly, $\tilde{q} = \begin{bmatrix}1 & 0\end{bmatrix}'$) yields the control law:

$$v(x) = \begin{cases} 99.75 & x_1(6.5x_1 + x_2) \le 0 \\ -99.75 & x_1(6.5x_1 + x_2) > 0. \end{cases}$$

3.0.4. Overall design algorithm

To summarize, we may design optimal rate controllers for a second order system of relative degree two of the form

$$P(s) = \frac{1}{s^2 + as + b}$$

by performing the following steps:

1. For a given gain bound v_0, check to make sure that the conditions
$$v_0 > \begin{cases} \left|\dfrac{a^2}{4} - b\right|, & a > 0 \\ \max\left\{\dfrac{a^2}{4} - b, b\right\}, & a \le 0 \end{cases}$$
and $v_0 > 2b - a^2/4$ are satisfied. If so, proceed to step 2.
2. Compute the eigenvectors of $A + v_0BC$ and the direction of the phase portraits of $\dot{x} = (A - v_0BC)x$ (clockwise or counterclockwise). Choose any vector q (and a corresponding choice of $\tilde{q}$) in the region where the phase portraits rotate in the same direction.
3. An optimal controller is given by
$$v(x) = \begin{cases} v_0 & x'\tilde{w}_s\tilde{q}'x \le 0 \\ -v_0 & x'\tilde{w}_s\tilde{q}'x > 0 \end{cases}$$
where $\tilde{w}_s$ and $\tilde{q}$ are normal vectors to w_s and q, respectively, that are oriented in the same direction.

4. Conclusion

We have provided an extension of the result in [11] to be able to design controllers with symmetric switching gains that optimize the rate of convergence of the state trajectories to the origin. Note that the controllers given here are not unique as the parameter q is a free parameter (subject to the constraints listed here). More detail on intelligent methods for choosing the parameter q can be found in Chapter 3 of [15].

The methods described herein have been extended in two different directions. First, in [13], the controllers we present here are shown to be *finite L2 gain stabilizing* in the presence of exogenous inputs/plant disturbances. Moreover, the results obtained in [13] hold not only when the true state is available, but they also hold when only an *estimate* of the state (produced via a standard Luenberger observer) is available. That these switching controllers can be shown to be finite L2 gain stabilizing has great consequences of both theoretical and practical concern, including stability in the presence of time delays. Furthermore, these results allow us to design switching controllers using the techniques described herein for classes of systems that can be nonlinear, time-varying, and/or of dimension greater than two which have a good second order LTI *approximant* in an L2 gain sense.

Clearly, the presence of an observer affects convergence rate (the rate of convergence of the state trajectory is affected by the rate at which the observer can converge to the true state), but in another document [14], we see that the presence of an observer does not inhibit this switching architecture from having practical uses. There, we examine the performance of the switching methods described here (upon which we expanded in [13]) for a class of step-tracking problems with objectives such as minimal overshoot and minimal settling time, and we compare these results with those obtained via more standard forms of LTI control.

As for future research directions, while [13] allows us to extend our results to some systems of higher dimension, it is in general desirable to examine whether/how the techniques described here and in [11] can be extended to LTI systems of arbitrary (but finite) dimension. Preliminary work indicates that an extension of the work shown here in which the stable eigenvector is replaced by a stable hyperplane, and the control is switched between multiple state feedback controllers achieves qualitative results that are similar to what we examine here.

As a final technical aside, the work presented here originally stemmed from the study of a similar problem involving the use of Pontryagin's Maximum Principle for a class of time-optimal control problems [12]. The reader interested in the historical formulation of this problem is invited to read this older manuscript, but the authors would like to explicitly point out that this formulation was only helpful in *discovering* the form of the optimal control considered here; the maximum principle has no real utility in establishing formal proofs for the convergence rate problems considered in this manuscript.

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