SANDIA REPORT

Mean Curl Formulation on Quadrilaterals with Application to Implicit Magnetics Diffusion Equations in Alegra 2D

John A. Mitchell
Multiphysics Simulation Technologies

Sandia National Laboratories

Mean Curl Formulation on Quadrilaterals
with
Application to Implicit Magnetics Diffusion Equations
in Alegra 2D

John A. Mitchell

Sandia National Laboratories

1515 Eubank SE

Albuquerque, NM 87185

Abstract

This report proposes a mean curl on quadrilateral elements for *Alegra 2D* magnetics calculations. The resulting quadrilateral element is an extension of the work [1, 4] to include the curl operator. The resulting quadrilateral element implementation is applicable to implicit calculations in Cartesian coordinates which use either the vector potential or *B* formulation. It is useful to note that in the vector potential formulation, the mean curl produces a mean flux density on the element, whereas in the *B* formulation the mean curl produces a mean current on the element. In both cases, the mean curl is, by construction, divergence free. The mean curl is further developed and applied to the nonlinear permeability capability that currently exists in *Alegra 2D* for the vector potential formulation. Using the proposed mean curl, a Jacobian operator is developed and presented which is applicable to nonlinear iterations associated with the implicit problem. Note that the existing nonlinear permeability does not include hysteresis; also, the current nonlinear implementation uses a Jacobian free solution algorithm because a Jacobian operator has neither been developed or implemented. A Jacobian for the full quadrature implementation that currently exists in *Alegra 2D* can be similarly derived based upon the results presented.

3

Acknowledgment

Contents

Appendix

5

List of Figures

List of Tables

1   Introduction and Motivation

This report proposes a mean curl on quadrilateral elements for *Alegra 2D* magnetics calculations. The resulting quadrilateral element implementation is applicable to implicit calculations in Cartesian coordinates which use either the vector potential or *B* formulation. It is useful to note that in the vector potential formulation, the mean curl produces a mean flux density on the element, whereas in the *B* formulation the mean curl produces a mean current on the element. In both cases, the mean curl is, by construction, divergence free. The mean curl is further developed and applied to the nonlinear permeability capability that currently exists in *Alegra 2D* for the vector potential formulation. Using the proposed mean curl, a Jacobian operator is developed and presented which is applicable to nonlinear iterations associated with the implicit problem. Note that the existing nonlinear permeability does not include hysteresis; also, the current nonlinear implementation uses a Jacobian free solution algorithm because a Jacobian operator has neither been developed or implemented. A Jacobian for the full quadrature implementation that currently exists in *Alegra 2D* can be similarly derived based upon the results presented.

The mean curl has two potential advantages over the existing *Alegra 2D* Gauss quadrature implementation. First, the mean curl allows for closed form evaluation of all quantities required and thus eliminates Gauss quadrature loops. For example, in the vector potential formulation, the action of the operator and calculation of the Jacobian matrix for Newton iterations, are simple, easily implemented formulas that can be directly evaluated thus yielding faster code. The second potential advantage relates to *Alegra* data structures that store material properties for an element. *Alegra* stores one value for a particular property on each element and does not facilitate storing values associated with each Gauss point on an element. This limitation may conflict with constitutive models which evaluate properties as a function of field values on an element. For example, in the vector potential formulation, nonlinear permeability models may need to evaluate the permeability as a function of the flux density which may vary on an element. By using the mean curl, data requirements for calculations are consistent with existing *Alegra* data structures. The mean curl yields a single, constant value for the flux density on an element which can then be used to evaluate a single nonlinear permeability value on the element.

An outline for the report is now given. Section 2 briefly introduces the magnetic diffusion equations that are considered, that is, Ampere's law, Faraday's law, Ohm's law and a magnetic constitutive model. Section 3 quickly develops the weak form for the vector potential formulation and then motivates and develops the core mean curl results which are applicable to both the vector potential formulation and *B* formulation. The weak form and application of the mean curl for the *B* formulation is presented in Section 4. The mean curl introduces spurious modes that must be stabilized. Stabilization of the mean curl is presented in Section 5. These results are also applicable to the nonlinear permeability Jacobian that is presented in Section 6.

2   Resistive Magnetic Diffusion Equations

Alegra 2D performs a *resistive magnetic diffusion* calculation. This calculation is the last phase in the operator split algorithm associated with the advancement by a single time step. The physics

of magnetic diffusion is assumed to satisfy magneto-static and electro-static approximations and is mathematically represented by a simplified form of Maxwell's equations. The magnetic diffusion equations solved in *Alegra 2D* are derived from Ampere's law and Faraday's law. Ohm's law, a constitutive model relating the magnetic field to the flux density is appended.

Ampere's Law

$$\nabla \times \underline{H} = \underline{J} \tag{1}$$

$\underline{H}$ and $\underline{J}$ are the magnetic field and current density respectively. The time rate of change of displacement currents in Ampere's law are ignored based upon the electro-static assumption.

Faraday's Law

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}, \tag{2}$$

$\underline{B}$ and $\underline{E}$ are the flux density and electric field respectively.

Ohm's Law

$$\underline{J} = \sigma \underline{E} \tag{3}$$

Ohm's law is a constitutive model relating the current density to the magnetic field through the material conductivity denoted by σ. In this report, the conductivity is taken as a constant. Notation for the resistivity $\eta = \frac{1}{\sigma}$ is also used.

Magnetic Constitutive Model

$$\underline{B} = \mu \underline{H} \tag{4}$$

Magnetic permeability is denoted by μ. In this report, μ may depend on the magnitude of the flux density $\|\underline{B}\|$ in a nonlinear but smooth way. However, hysteresis and history effects are not part of the nonlinearities considered here. This assumption on the constitutive model may not be physically correct for some materials but was assumed as a starting point for this project work. This relation is also assumed to be invertible so that the following makes sense,

$$\underline{H} \;=\; \frac{\underline{B}}{\mu(\|\underline{B}\|)} = \nu(\|\underline{B}\|)\underline{B}, \tag{5}$$

where $\nu(\|\underline{B}\|)$ denotes the material reluctivity.

3 *Alegra 2D*: Vector Potential Formulation

In the vector potential formulation [3], the flux density is taken as:

$$\underline{B} = \nabla \times \underline{A}, \tag{6}$$

where $\underline{B}$ has two components in the plane spanned by unit vectors $(\hat{\imath}, \hat{\jmath})$, and $\underline{A} = A\hat{k}$ is the vector potential. Note the unit vector $\hat{k} = \hat{\imath} \times \hat{\jmath}$. Under this assumption, the divergence of the flux density is zero, and from Faraday's law (2),

$$\nabla \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t} \right) = 0 \implies \underline{E} = -\frac{\partial \underline{A}}{\partial t} + \underline{E}_0, \tag{7}$$

where $\underline{E}_0$ is the gradient of a scalar potential. This latter condition is given because the curl of the gradient is zero.

Amperes law (1) is written using (6) and the constitutive model (4).

$$\nabla \times \left(\frac{\nabla \times \underline{A}}{\mu} \right) = \underline{J} \tag{8}$$

The above equation is combined with Ohm's law (3) and (7) to arrive at the governing equation for the vector potential formulation.

$$\nabla \times \left(\frac{\nabla \times \underline{A}}{\mu} \right) = -\sigma \frac{\partial \underline{A}}{\partial t} + \sigma \underline{E}_0 \tag{9}$$

Boundary conditions will be introduced in the next section.

3.1 Weak form

To arrive at the weak form for (9), dot (9) with a vector function $\underline{\hat{E}}$, and integrate over the domain Ω.

$$-\int \sigma \left[\underline{\hat{E}} \cdot \frac{\partial \underline{A}}{\partial t} - \underline{\hat{E}} \cdot \underline{E}_0 \right] d\Omega \;=\; \int \underline{\hat{E}} \cdot \left[\nabla \times \left(\frac{\nabla \times \underline{A}}{\mu(\|\underline{B}\|)} \right) \right] d\Omega \tag{10}$$

The vector identity $\nabla \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\nabla \times \underline{a}) - \underline{a} \cdot (\nabla \times \underline{b})$, with $\underline{a} = \frac{\nabla \times \underline{A}}{\mu(\|\underline{B}\|)}$ and $\underline{b} = \underline{\hat{E}}$ is useful for deriving the boundary conditions and weak form associated with the integral on the right hand

side. This is given as

$$\int \underline{\hat{E}} \cdot \left[\nabla \times \left(\frac{\nabla \times \underline{A}}{\mu(\|\underline{B}\|)} \right) \right] d\Omega = \int \nabla \cdot \left[\left(\frac{\nabla \times \underline{A}}{\mu(\|\underline{B}\|)} \right) \times \underline{\hat{E}} \right] d\Omega$$

$$+ \int \frac{1}{\mu(\|\underline{B}\|)} (\nabla \times \underline{A}) \cdot (\nabla \times \underline{\hat{E}}) \, d\Omega$$

$$= L_{\underline{\hat{E}}} + \int \frac{1}{\mu(\|\underline{B}\|)} (\nabla \times \underline{A}) \cdot (\nabla \times \underline{\hat{E}}) \, d\Omega \tag{11}$$

where $L_{\underline{\hat{E}}}$, after applying the divergence theorem, gives the boundary conditions. Combining the last equation in (11) with (10) gives an electromagnetic energy equation for the domain. This is also the weak form for this formulation.

Weak form:

$$-\int \sigma \left[\underline{\hat{E}} \cdot \frac{\partial \underline{A}}{\partial t} - \underline{\hat{E}} \cdot \underline{E}_0 \right] d\Omega = \int \frac{1}{\mu(\|\underline{B}\|)} (\nabla \times \underline{A}) \cdot (\nabla \times \underline{\hat{E}}) \, d\Omega + L_{\underline{\hat{E}}} \tag{12}$$

The left hand side represents the rate of work done by electromagnetic fields due to motion of charge. The integral on the right hand side is the rate of energy stored in the magnetic field and $L_{\underline{\hat{E}}}$ is apparently the rate of energy flow out of the domain Ω through its boundary $\partial\Omega$. Boundary conditions are given by applying the divergence theorem to $L_{\underline{\hat{E}}}$ (defined in (11)).

$$L_{\underline{\hat{E}}} = \int \nabla \cdot \left[\left(\frac{\nabla \times \underline{A}}{\mu(\|\underline{B}\|)} \right) \times \underline{\hat{E}} \right] d\Omega = \int_{\partial\Omega} \hat{n} \cdot \left[\left(\frac{\nabla \times \underline{A}}{\mu(\|\underline{B}\|)} \right) \times \underline{\hat{E}} \right] d\partial\Omega, \tag{13}$$

where $\partial\Omega$ denotes the boundary of the domain and $\hat{n} = n_x \hat{i} + n_y \hat{j}$ denotes the normal to the boundary. The second form of the above is used to define Dirichlet and Neuman boundary conditions by identifying the magnetic field $\underline{H} = \frac{\nabla \times \underline{A}}{\mu(\|\underline{B}\|)}$, and using permutation properties of the scalar triple product.

Dirichlet BC: The weight function $\underline{\hat{E}}$ is a surrogate for the electric field. The following equation is a natural permutation of (13) and is a form convenient for this specification, where the term $\hat{n} \times \underline{H}$ is in the $\hat{k}$ direction, that is, normal to the computational plane.

$$L_{\underline{\hat{E}}} = \int_{\partial\Omega} \underline{\hat{E}} \cdot [\hat{n} \times \underline{H}] \, d\partial\Omega, \tag{14}$$

Therefore, because of the dot product, the Dirichlet condition is on the z component of the electric field.

Neuman BC: The tangential component of the magnetic field $\underline{H}$ is specified on the boundary $\partial\Omega$. This follows directly from (14) and corresponds with the H tangent boundary condition in *Alegra 2D*. If a non-homogeneous value is not specified, then a zero value for the tangential

12

magnetic field is weakly enforced on the boundary.

3.2 Mean curl on quadrilateral elements, Cartesian coordinates

Spatial material properties in *Alegra 2D*, such as permeability μ and conductivity σ, have a single value on each element which is assumed to be constant over the entire element. Depending upon the functional form of the permeability $\mu = \mu(\|\underline{B}\|)$, this assumption may or may not be true because $\underline{B}$ can vary on an element. For example, when evaluating the right hand side of (12), where the weight function $\underline{\hat{E}}$ takes on values associated with linear basis functions, components of $\underline{B}$ will vary linearly. It follows that the quadrature algorithm used to evaluate (12) must evaluate $\mu = \mu(\|\underline{B}\|)$ at each quadrature point. However, data structures in *Alegra 2D* only support a single cell quantity for μ on the element. This limitation is the main motivation for the development of a mean curl on the element. The remainder of this section develops the mean curl on a quadrilateral element.

Evaluation of the integral on the right hand side of (12) is considered. On an element, the key step and assumption is to use a mean value for the flux density when evaluating this integral. The mean value for the flux density on an element is denoted and defined as:

$$\underline{\overline{B}} = \overline{\nabla \times \underline{A}} = \frac{\int_e (\nabla \times \underline{A}) \, d\Omega_e}{\int_e d\Omega_e}. \tag{15}$$

Using $\underline{\overline{B}}$, a single value for the permeability can be evaluated for the element. Furthermore, since the flux density is taken as constant, it can be pulled outside the integral. Below, the reluctivity $\nu(\|\underline{\overline{B}}\|)$ (see (5)) is used to denote the reciprocal of the permeability and its dependence on the mean value of the magnitude of the flux density. The approximated version of the integral on the right hand side of (12) is denoted by I:

$$I = \nu(\|\underline{\overline{B}}\|)\underline{\overline{B}} \cdot \int_e (\nabla \times \underline{\hat{E}}) \, d\Omega_e. \tag{16}$$

To proceed further, the finite element discretization is introduced. For mean curl calculations, it is convenient to represent the quadrilateral basis functions on a reference element, shown in Figure 1, using the following set of functions ϕ_i,

$$\phi_i = \frac{1}{4}\Sigma_i + \frac{1}{2}\xi\Lambda_{1i} + \frac{1}{2}\eta\Lambda_{2i} + \xi\eta\Gamma_i \tag{17}$$

where local coordinates on the reference element are denoted by ξ and η, i represents the local node index which takes on values of $i \in (0, 1, 2, 3)$, and components Σ_i, Λ_{1i}, Λ_{2i} and Γ_i of the quadrilateral base vectors are given in Table 1. On an element, the vector potential function $\underline{A} = A\hat{k}$ is interpolated with the above nodal basis functions in the usual way:

$$\underline{A} = A\hat{k} = (\sum_j A_j\phi_j)\hat{k} \triangleq \sum_j A_j\underline{E}_j \tag{18}$$
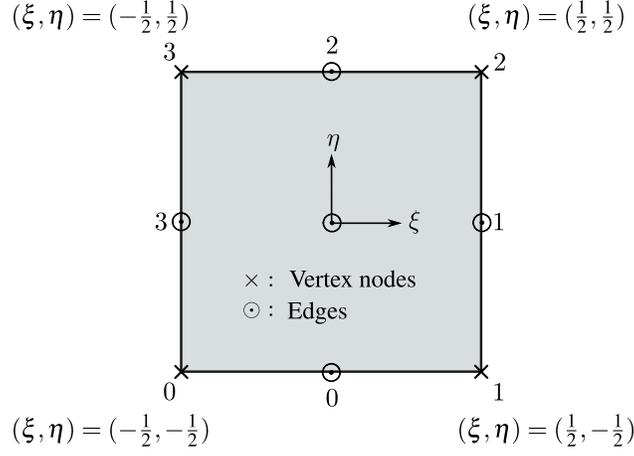
Figure 1. Quadrilateral reference element

where A without an underline denotes the single unknown component of the vector potential, A_j denotes nodal values of A, and $\hat{k}$ denotes the unit vector normal to the computational domain Ω.

Table 1. Components of Quadrilateral Base Vectors

Node i	ξ	η	Σ	Λ_1	Λ_2	Γ
0	-0.5	-0.5	1	-1	-1	1
1	0.5	-0.5	1	1	-1	-1
2	0.5	0.5	1	1	1	1
3	-0.5	0.5	1	-1	1	-1

The weight function $\hat{\underline{E}}$ in (16) is replaced with $\underline{E}_i$ (see (18)) which defines an element contribution from local node i as

$$I_i = \nu(\|\overline{\underline{B}}\|)\overline{\underline{B}} \cdot \int_e (\nabla \times \underline{E}_i)\, d\Omega_e. \tag{19}$$

The approach proposed here is equivalent to evaluation of the integral on the right hand side of (12) using one-point quadrature. The downside of one-point quadrature is one spurious (zero-energy) mode is introduced. There are two vectors in the null space of the element level operator implied by (19). The first corresponds with constant values of the vector potential on an element. This vector is denoted by Σ and it does not present any problems as the finite element approximation is expected to capture constant values. Uniqueness with this mode is typically resolved via dirichlet boundary conditions or for transient calculations through enhancement of the definiteness via the mass matrix. The second vector is the spurious hourglass mode Γ. Components of incoming nodal values for the vector potential that are aligned with Γ do not contribute to nodal contributions in (19), i.e., although a non-trivial value for the vector potential and its curl exist on the element, $I_i = 0$ for each i. Fortunately, stabilization of the hourglass mode is easily accomplished using the orthogonal properties of the base vectors. Components for Σ and Γ are given in Table 1.

14

Before jumping into the details a brief outline of the strategy and goal is given. The principal goal is to demonstrate that I_i in (19) is a linear combination of the base vector components Λ_{1i} and Λ_{2i}. Then stabilization can be accomplished in two different but equivalent ways. In the first case, residual calculations are stabilized by extracting the component of the incoming vector potential that is along Γ and adding a small contribution along Γ that is proportional to $\nu(\|\overline{\underline{B}}\|)$. The second form of stabilization occurs when forming element matrices implied by (19). It is shown that element matrices are a linear combination of tensor products of the base vectors Λ_1 and Λ_2. Then an equivalent stabilization can be implemented by adding contribution to the stiffness matrix that is along the tensor product of Γ with itself. These details are now presented. Note that it is only necessary to work through these details to demonstrate these properties. Calculations using the stabilized mean curl are very straightforward and do not necessarily require all these results.

To begin, the integral in (19) is written as:

$$\int_e (\nabla \times \underline{E}_i)\, d\Omega_e \;=\; \overline{\nabla \times \underline{E}_i}\, C_e, \tag{20}$$

where C_e denotes the area of element e

$$C_e \;=\; \int_e d\Omega_e.$$

Given the mean curl in (20), the mean value of the flux density on an element can be directly evaluated:

$$\begin{aligned}
\overline{\underline{B}} \;&=\; \overline{\nabla \times \underline{A}} \\
&=\; \sum_{j=0}^{4} A_j (\overline{\nabla \times \underline{E}_j}).
\end{aligned} \tag{21}$$

From the above, entries k_{ij} in an element stiffness matrix are given by:

$$k_{ij} = \nu(\|\overline{\underline{B}}\|) \left[(\overline{\nabla \times \underline{E}_i}) \cdot (\overline{\nabla \times \underline{E}_j}) \right] C_e. \tag{22}$$

As denoted, the above relation may be nonlinear if the reluctivity depends upon $\|\overline{\underline{B}}\|$. This nonlinearity has no effect on the mean curl evaluation.

The focus is now on evaluation of $\overline{\nabla \times \underline{E}_j}$. To begin, Green's theorem is applied to (20):

$$\int_e (\nabla \times \underline{E}_i)\, d\Omega_e = - \oint_e \phi_i(\xi,\eta) \left[\underline{u}_\xi(\xi,\eta)\, d\xi + \underline{u}_\eta(\xi,\eta)\, d\eta \right], \tag{23}$$

where

$$\underline{u}_\xi(\xi,\eta) = \frac{\partial x}{\partial \xi}\hat{i} + \frac{\partial y}{\partial \xi}\hat{j}, \text{ and } \underline{u}_\eta(\xi,\eta) = \frac{\partial x}{\partial \eta}\hat{i} + \frac{\partial y}{\partial \eta}\hat{j}. \tag{24}$$

The integral on the right hand side is a path integral over the boundary of the reference element (see Figure 1). On the edges of a linear quadrilateral element, values of $\underline{u}_\xi$ and $\underline{u}_\eta$ are either constant

and non-zero or zero. This simplifies the integral substantially:

$$\int_e (\nabla \times \underline{E}_i)\, d\Omega_e = -\frac{1}{4}\left[\underbrace{(\underline{u}_\xi^0 + \underline{u}_\eta^1 - \underline{u}_\xi^2 - \underline{u}_\eta^3)}_{0}\Sigma_i + (\underline{u}_\eta^1 + \underline{u}_\eta^3)\Lambda_{1i} + (-\underline{u}_\xi^0 - \underline{u}_\xi^2)\Lambda_{2i} \right], \qquad (25)$$

where the quantity with the under brace is zero, and the vectors $\underline{u}_\xi^0, \underline{u}_\eta^1, \underline{u}_\xi^2, \underline{u}_\eta^3$ denote values for $\underline{u}_\xi$ and $\underline{u}_\eta$ on edges, with superscripts denoting the edge, as indicated in Figure 1. Therefore the curl on an element using one point quadrature is

$$\int_e (\nabla \times \underline{E}_i)\, d\Omega_e = -\frac{1}{4}\left[(\underline{u}_\eta^1 + \underline{u}_\eta^3)\Lambda_{1i} + (-\underline{u}_\xi^0 - \underline{u}_\xi^2)\Lambda_{2i} \right]. \qquad (26)$$

Using this result, a slightly different view of the element stiffness matrix (22) is written down. An operator K^e, representing the element stiffness matrix is given by

$$K^e = \frac{\nu(\|\underline{\overline{B}}\|)C_e}{16}\left[\alpha_{11}^e(\Lambda_1 \otimes \Lambda_1) + \alpha_{22}^e(\Lambda_2 \otimes \Lambda_2) - \alpha_{12}^e(\Lambda_1 \otimes \Lambda_2) - \alpha_{12}^e(\Lambda_2 \otimes \Lambda_1) \right] \qquad (27)$$

where the scalars α_{ij}^e are given as:

$$\begin{aligned}
\alpha_{11}^e &= (\underline{u}_\eta^1 + \underline{u}_\eta^3)\cdot(\underline{u}_\eta^1 + \underline{u}_\eta^3), \\
\alpha_{12}^e &= (\underline{u}_\eta^1 + \underline{u}_\eta^3)\cdot(\underline{u}_\xi^0 + \underline{u}_\xi^2), \\
\alpha_{22}^e &= (\underline{u}_\xi^0 + \underline{u}_\xi^2)\cdot(\underline{u}_\xi^0 + \underline{u}_\xi^2).
\end{aligned} \qquad (28)$$

For clarity, the tensor products above produce linear operators that act on vectors to produce another vector. Suppose $\underline{w} = \Lambda_1$ and consider the action of $(\Lambda_1 \otimes \Lambda_2)$ on $\underline{w}$.

$$\begin{aligned}
(\Lambda_1 \otimes \Lambda_2)\underline{w} &= (\Lambda_1 \otimes \Lambda_2)\Lambda_1 \\
&= \Lambda_1(\Lambda_2 \cdot \Lambda_1) \\
&= \Lambda_1(0) \\
&= 0
\end{aligned} \qquad (29)$$

Using the orthogonality properties of the base vectors in Table 1, it is easy to see that both Σ and Γ are in the null space of K^e, i.e., $K^e\Sigma = K^e\Gamma = \underline{0}$, where $\underline{0}$ is the zero vector.

3.3 Mean curl formulae

In this section, all the expressions required to implement the mean curl in a computer program are given. A sample implementation of the element stiffness matrix and its stabilization (described in Section 5) is given in the appendix.

The area C_e of a quadrilateral with nodal coordinates (x_i, y_i), for $j \in (0, 1, 2, 3)$ is

$$C_e = \int_e d\Omega_e = \frac{1}{2}((x_3 - x_1)(y_0 - y_2) + (x_0 - x_2)(y_1 - y_3)). \tag{30}$$

Components of the mean curl of $\underline{E}_i$ (see (18)) are given by

$$\int_e (\nabla \times \underline{E}_i) d\Omega_e = \overline{\nabla \times \underline{E}_i} C_e \Longrightarrow \frac{1}{2} \left\{ \begin{array}{l} (x_3 - x_1), \ (y_3 - y_1) \\ (x_0 - x_2), \ (y_0 - y_2) \\ (x_1 - x_3), \ (y_1 - y_3) \\ (x_2 - x_0), \ (y_2 - y_0) \end{array} \right\}. \tag{31}$$

Given nodal values A_j for the vector potential, components of the mean flux on an element (see (21)) are

$$\underline{B} = \overline{\nabla \times \underline{A}} \Longrightarrow \frac{1}{2 C_e} \left\{ \begin{array}{l} (A_1 - A_3)(x_0 - x_2) - (A_0 - A_2)(x_1 - x_3) \\ (A_1 - A_3)(y_0 - y_2) - (A_0 - A_2)(y_1 - y_3) \end{array} \right\}. \tag{32}$$

The element contribution to node i can be evaluated directly from (19) using the above formulae.

$$I_i = \nu(\|\underline{\overline{B}}\|) \underline{\overline{B}} \cdot \int_e (\nabla \times \underline{E}_i) d\Omega_e$$

For practical reasons, it makes sense to compute components of the curl of $\underline{E}_i$ using (31), as well as the mean flux (32), and then perform a dot product between them to compute components I_i.

There are a variety of ways to compute entries in the element stiffness matrix (22). One approach is to form the tensor product using (31). Whatever the method, the unique entries of the symmetric element stiffness matrix are given as

$$K^e \Longrightarrow \frac{\nu(\|\underline{\overline{B}}\|)}{4 C_e} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix}, \tag{33}$$

where the unique components k_{ij} are given by

$$
\begin{aligned}
k_{11} &= (x_1 - x_3)^2 + (y_1 - y_3)^2 \\
k_{12} &= (x_0 - x_2)(x_3 - x_1) + (y_0 - y_2)(y_3 - y_1) \\
k_{13} &= -(x_1 - x_3)^2 - (y_1 - y_3)^2 \\
k_{14} &= (x_0 - x_2)(x_1 - x_3) + (y_0 - y_2)(y_1 - y_3) \\
k_{22} &= (x_0 - x_2)^2 + (y_0 - y_2)^2 \\
k_{23} &= (x_0 - x_2)(x_1 - x_3) + (y_0 - y_2)(y_1 - x_3) \\
k_{24} &= -(x_0 - x_2)^2 - (y_0 - y_2)^2 \\
k_{33} &= (x_1 - x_3)^2 + (y_1 - x_3)^2 \\
k_{34} &= (x_2 - x_0)(x_1 - x_3) + (y_2 - y_0)(y_1 - x_3) \\
k_{44} &= (x_0 - x_2)^2 + (y_0 - y_2)^2.
\end{aligned}
\tag{34}
$$

4 *Alegra 2D*: B Formulation

4.1 Weak form

In the *Alegra 2D B*-formulation [3], the flux density has only one component which is normal to the computational domain Ω. As a vector field, the flux density is denoted as $\underline{B} = B\hat{k}$, where the component B is the primary unknown in the problem. One limitation of the B-formulation in *Alegra 2D* is that the material permeability μ must be taken as constant; therefore it is not possible to model problems that require nonlinear permeability with the B-formulation.

To arrive at the weak form, dot Faraday's law (2) with a vector function $\hat{\underline{H}}$, and integrate over the domain Ω; use the vector identity $\hat{\underline{H}} \cdot (\nabla \times \underline{E}) = \nabla \cdot (\underline{E} \times \hat{\underline{H}}) + \underline{E} \cdot (\nabla \times \hat{\underline{H}})$. Then, Ampere's law (1) combined with Ohm's law (3) and the magnetic constitutive model (5), is used on the second term in the vector identity. Note that since μ is taken as constant, $\nabla \times \underline{H} = \frac{1}{\mu} \nabla \times \underline{B}$. These steps are given below.

$$
\begin{aligned}
-\int \hat{\underline{H}} \cdot \frac{\partial \underline{B}}{\partial t} \, d\Omega &= \int \hat{\underline{H}} \cdot (\nabla \times \underline{E}) \, d\Omega \\
&= \int \left[\nabla \cdot (\underline{E} \times \hat{\underline{H}}) + \underline{E} \cdot (\nabla \times \hat{\underline{H}}) \right] d\Omega \\
&= L_{\hat{\underline{H}}} + \int \left[\underline{E} \cdot (\nabla \times \hat{\underline{H}}) \right] d\Omega \\
&= L_{\hat{\underline{H}}} + \int \left[\frac{(\nabla \times \underline{B})}{\sigma\mu} \cdot (\nabla \times \hat{\underline{H}}) \right] d\Omega
\end{aligned}
$$

where $L_{\hat{\underline{H}}}$, after applying the divergence theorem, gives boundary conditions. These expressions are the weak form for the B-formulation.

18

Weak form:

$$-\int \hat{\underline{H}} \cdot \frac{\partial \underline{B}}{\partial t} d\Omega = \int \left[\frac{(\nabla \times \underline{B})}{\sigma \mu} \cdot (\nabla \times \hat{\underline{H}}) \right] d\Omega + L_{\hat{\underline{H}}} \tag{35}$$

Boundary conditions in the *B*-formulation are given by applying the divergence theorem to $L_{\hat{\underline{H}}}$ and are conceptually similar to those of the vector potential formulation.

$$L_{\hat{\underline{H}}} = \int \nabla \cdot (\underline{E} \times \hat{\underline{H}}) d\Omega = \int_{\partial \Omega} \hat{n} \cdot (\underline{E} \times \hat{\underline{H}}) d\partial \Omega, \tag{36}$$

where $\partial \Omega$ denotes the boundary of the domain and $\hat{n} = n_x \hat{i} + n_y \hat{j}$ denotes the boundary normal. The boundary integral on the right is used to define Dirichlet and Neuman conditions using properties of the scalar triple product to permute terms.

Dirichlet BC: The weight function $\hat{\underline{H}}$ is a surrogate for the magnetic field. $\hat{\underline{H}}$ is specified on $\partial \Omega$ and is normal to the 2*D* domain Ω. The following equation is a natural permutation of (36) and is a form convenient for this specification, where the term $\hat{n} \times \underline{E}$ is in the $\hat{k}$ direction, that is, normal to the computational plane.

$$L_{\hat{\underline{E}}} = \int_{\partial \Omega} \hat{\underline{H}} \cdot [\hat{n} \times \underline{E}] d\partial \Omega, \tag{37}$$

Therefore, because of the dot product, the Dirichlet condition is on the *z* component of the magnetic field.

Neuman BC: The tangential component of the magnetic field $\underline{E}$ is specified on the boundary $\partial \Omega$. This follows directly from (37) and corresponds with the *E* tangent boundary condition in *Alegra 2D*. If a non-homogeneous value is not specified, then a zero value for the tangential electric field is weakly enforced.

4.2 Mean curl on quadrilateral elements, Cartesian coordinates

From the mean curl point of view, this section is nearly identical to Section 3.2. There, the mean curl was used to calculate the mean flux density on an element. In this section, the mean curl is used to calculate the mean current on an element.

Evaluation of the integral on the right hand side of (35) is considered. On an element, the key step and assumption is to use the mean value of the current when evaluating this integral.

The mean current $\overline{\underline{J}}$ is defined by:

$$\overline{\underline{J}} = \frac{\int_e \frac{(\nabla \times \underline{B})}{\mu} d\Omega_e}{\int_e d\Omega_e}. \tag{38}$$

Given the mean current $\overline{\underline{J}}$, the approximated version of the integral on the right hand side of (35) is denoted by V:

$$V = \eta \overline{\underline{J}} \cdot \int_e (\nabla \times \hat{\underline{H}}) d\Omega_e, \tag{39}$$

where η denotes material resistivity (see (3)). For simplicity, dependence of η on the mean current was not denoted here although that could be included. Here the resistivity (reciprocal of the conductivity) is assumed to be constant. This is not an impediment; perhaps the conductivity depends upon the current through temperature in which case the mean value of the current can be used to calculate the resistivity.

To proceed further, the finite element discretization is introduced, where the previously defined basis functions (17) on a quadrilateral are used. On an element, the flux density $\underline{B}$ is interpolated with the basis functions in the usual way:

$$\underline{B} = B\hat{k} = \left(\sum_j B_j \phi_j\right)\hat{k} \triangleq \sum_j B_j \underline{H}_j \tag{40}$$

where B without an underline denotes the single unknown component of the flux density, B_j denotes nodal values of B, and $\hat{k}$ denotes the unit vector normal to the computational domain Ω. The weight function $\hat{\underline{H}}$ in (35) is replaced with $\underline{H}_i$ (see (40)) which defines an element contribution from local node i as

$$V_i = \eta \overline{\underline{J}} \cdot \int_e (\nabla \times \underline{H}_i) d\Omega_e. \tag{41}$$

At this stage, nearly all the results and analysis from the vector formulation can be applied. Those results are given in the following section.

4.3   Mean curl formulae

In this section, formulas relating to the mean curl for the B formulation are given. Only those that are unique and or distinct from the vector potential formulation are included, otherwise a reference to the equation number is given. A sample computer implementation of the element stiffness matrix and its stabilization (described in Section 5) is given in the appendix.

The cell area of a quadrilateral C_e, and the mean curl of shape functions $\overline{\nabla \times \underline{H}_i}$ are given in (30) and (31) respectively, with $\underline{E}_i$ replaced with $\underline{H}_i$ in the latter.

The mean current $\overline{\underline{J}}$ is analogous to the mean flux density. Using the definition (38) and the previous

20

result (32), this is written as

$$\underline{\bar{J}} = \frac{\overline{\nabla \times \underline{B}}}{\mu} \Longrightarrow \frac{1}{2C_e\mu} \left\{ \begin{array}{c} (B_1-B_3)(x_0-x_2)-(B_0-B_2)(x_1-x_3) \\ (B_1-B_3)(y_0-y_2)-(B_0-B_2)(y_1-y_3) \end{array} \right\}. \tag{42}$$

For practical reasons, it makes sense to compute components of the curl of $\underline{H}_i$ using (31), as well as the mean current (42), and then perform a dot product between them to compute components V_i in (41).

An element stiffness matrix analogous to (33) is given as

$$K^e \Longrightarrow \frac{\eta}{4C_e\mu} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix}, \tag{43}$$

where the unique components k_{ij} are identical to those given in (34).

5   Stabilization

As described in Section 3.2, the mean curl approach introduces spurious modes. If these modes are not stabilized, then the mean curl approach is useless. This is the how-to section for stabilizing the mean curl quadrature described in this report. Stabilization can be accomplished in two different but equivalent ways. Depending upon the context, it may be necessary to implement both.

In the first case, action of the mean curl operator is evaluated without use of the element stiffness matrix. This operator may correspond with either (19) or (41) for either the vector potential or B-formulation respectively. In both cases, an incoming vector of nodal quantities associated with an element, either (A_0, A_1, A_2, A_3) or (B_0, B_1, B_2, B_3) are given. Whichever the case, let the given components define a vector $\underline{u}$. The basic idea is to compute a new vector $\underline{u}_\Gamma = (\underline{u} \cdot \Gamma)\Gamma$ which is a projection of $\underline{u}$ along the hourglass vector Γ. The output of the operator is augmented/summed with $\alpha \underline{u}_\Gamma$, where α is a small number proportional to the appropriate material properties.

The second form of stabilization is implemented by stabilizing the element stiffness matrix. This is accomplished by augmenting K^e, (33) or (43), for either the vector potential formulation or the B-formulation respectively. In both cases, the element stiffness matrix is stabilized by adding a matrix $K_\Gamma^e = \alpha \Gamma \otimes \Gamma$.

Recall that the mean curl operator produces the zero vector for any incoming fields along the hourglass vector. Then these methods artificially eliminate the null space of the operator that is associated with Γ. The following vectors will stabilize these operators when they are used in either linear or nonlinear solution algorithms. Note that the scaling is arbitrary. Material properties are introduced to make the ficticious stabilization vectors scale in a way that may be meaningful for the physics.

Stabilizing the vector potential formulation The operator (19) defining I_i is augmented with I_{Γ_i}.

$$I_i \Longrightarrow I_i + I_{\Gamma_i} \tag{44}$$

where

$$I_{\Gamma_i} = \alpha \frac{\nu(\|\overline{B}\|)}{2C_e}(A_0 - A_1 + A_2 - A_3)\Gamma_i,$$

$\alpha > 0$ is a small, user specified stabilizing constant, and Γ_i is a component of the hourglass mode defined in Table 1. The stiffness matrix (33) associated with the vector potential formulation is augmented with the K_Γ^e.

$$K^e \Longrightarrow K^e + K_\Gamma^e, \tag{45}$$

where

$$K_\Gamma^e \Longrightarrow \alpha \frac{\nu(\|\overline{B}\|)}{4C_e} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

Stabilizing the B-formulation The operator (41) defining V_i is augmented with V_{Γ_i}.

$$V_i \Longrightarrow V_i + V_{\Gamma_i} \tag{46}$$

where

$$V_{\Gamma_i} = \alpha \frac{\eta}{2C_e\mu}(B_0 - B_1 + B_2 - B_3)\Gamma_i,$$

$\alpha > 0$ is a small, user specified stabilizing constant, and Γ_i is a component of the hourglass mode defined in Table 1. The stiffness matrix (43) associated with the vector potential formulation is augmented with the K_Γ^e.

$$K^e \Longrightarrow K^e + K_\Gamma^e, \tag{47}$$

where

$$K_\Gamma^e = \alpha \frac{\eta}{4C_e\mu} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

6 Linearization of the Vector Potential Formulation for Non-linear Permeability

This section is only relevant to the vector potential formulation using the mean curl. A similar result can also be obtained for full quadrature but that is not presented.

For nonlinear implicit calculations, a Jacobian matrix is needed for Newton-Raphson iterations. Nonlinearities are assumed to be smooth, i.e., the reluctivity $v = v(\|\underline{B}\|)$, see (5), is assumed to be differentiable with respect to the magnitude of the magnetic field $\|\underline{B}\|$. Hysteresis and history effects are not part of the nonlinearities considered here. If these assumptions on the reluctivity are met, then the exact form and dependence of μ on the flux density is not critical, that is, it should be possible to derive the required derivative for other forms.

The element Jacobian matrix is calculated by differentiating (19) with respect to nodal quantities A_k. Denote the Jacobian as $K_{ik} = \frac{\partial I_i}{\partial A_k}$. Using the chain rule,

$$K_{ik} = \frac{\partial v}{\partial \|\underline{B}\|} \frac{\partial \|\overline{\underline{B}}\|}{\partial \overline{B}_j} \frac{\partial \overline{B}_j}{\partial A_k} \overline{\underline{B}} \cdot \int_e (\nabla \times \underline{N}_i) \, d\Omega_e + v(\|\overline{\underline{B}}\|) \frac{\partial \overline{\underline{B}}}{\partial A_k} \cdot \int_e (\nabla \times \underline{N}_i) \, d\Omega_e.$$

Note that

$$\frac{\partial \|\overline{\underline{B}}\|}{\partial \overline{B}_j} \frac{\partial \overline{B}_j}{\partial A_k} = \hat{n}_{\overline{B}} \cdot (\overline{\nabla \times \underline{N}_k}), \quad \text{where } \hat{n}_{\overline{B}} = \frac{\overline{\underline{B}}}{\|\overline{\underline{B}}\|}.$$

Then the Jacobian is written as

$$K_{ik} = v_T \, \hat{n}_{\overline{B}} \cdot (\overline{\nabla \times \underline{N}_k}) \, \overline{\underline{B}} \cdot \int_e (\nabla \times \underline{N}_i) \, d\Omega_e + v(\|\overline{\underline{B}}\|) \, \overline{\nabla \times \underline{N}_k} \cdot \int_e (\nabla \times \underline{N}_i) \, d\Omega_e.$$

where

$$v_T = \frac{\partial v}{\partial \|\overline{\underline{B}}\|}.$$

The Jacobian represents a change in I_i (see (19)) at each nodal point that arises due to small changes in the vector potential. To evaluate the above on an element, nodal quantities (A_0, A_1, A_2, A_3) are expected to be known. With these values, the mean flux density $\|\overline{\underline{B}}\|$ can be evaluated from the mean curl of the vector potential on the element thus allowing evaluation of the reluctivity $v(\|\overline{\underline{B}}\|)$. Similarly, it must then be possible to evaluate the slope of the reluctivity curve v_T at the computed mean value of the flux density $\|\overline{\underline{B}}\|$ on the element.

The above Jacobian includes two terms. The first term is a constitutive model response and is due to a change in the reluctivity. The second term on the right is a direct change due to change in the vector potential; this term is a direct effect resulting from the fact that the mean magnetic field is changing. Except for v_T, which is a material dependent property that must be given, all of the terms required to compute the above Jacobian were given in Section 3.3.

The above linear operator can be stabilized using the previously defined stabilization (45); it has exactly the same null space as the previously defined element stiffness matrix (22). Note that

the second term in the Jacobian is identical to (22). The first term is the tensor product of a two vectors, $\underline{u}_k = \hat{n}_{\overline{\underline{B}}} \cdot (\overline{\nabla \times \underline{N}_k})$ and $\underline{v}_i = \overline{\underline{B}} \cdot (\overline{\nabla \times \underline{N}_i}) C_e$. Both $\underline{u}_k$ and $\underline{v}_i$ are a linear combination of the components of Λ_1 and Λ_2 and therefore it follows that the tensor product will be a linear combination of them as well.

7 Demonstration Calculations on a Single Element

In this section, an eigen analysis is performed on a typical element stiffness matrix computed using the mean curl. The element stiffness matrix is subsequently stabilized and the eigen analysis is repeated. With this approach, element stiffness matrix properties described earlier in the report are easily demonstrated in a practical setting. Results presented are typical and representative of what would be seen in the practical application of this element.

The element used for demonstration calculations is depicted in Figure 2, where nodal coordinates (x, y) of each vertex are denoted. The element stiffness matrix for the vector potential formulation or the B-formulation, given by (33) or (43) is computed assuming that the material property (either $\frac{\eta}{\mu} = 1.0$ or $\nu(\|\overline{\underline{B}}\|) = 1.0$) on the element. This assumption is only for simplicity and is not a limitation or restriction on the relevance of the demonstration calculations. Several key points are illustrated.

- Observe that there are 2 zero eigenvalues for the case without stabilization.

- Only one eigenvalue is changed after stabilization, i.e., the eigenvalue associated with the hourglass mode is now positive; the remaining eigenvalues are unchanged.

- After stabilization (see (45) or (47)), the eigenvalue that remains zero is associated with the eigen vector representing constant solutions on the element. This is a requirement for the element to properly represent the constant solution.

- Observe that after stabilization, the eigen vectors associated with the original non-zero eigen values are unchanged from the un-stabilized matrix.

Listed below are results of the eigenvalue analysis with stabilization ($\alpha = 10^{-4}$) and without stabilization ($\alpha = 0$). It is illustrative to consider the action of the element operator on the hourglass vector in both cases. Because the material property on the element is taken as unity, this produces a vector which is proportional to the stabilization parameter and inversely proportional the element area C_e. See (33) and (43).
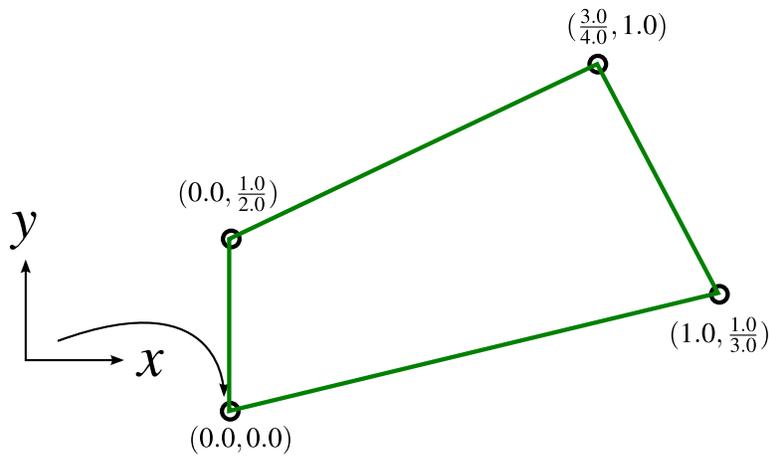
Figure 2. Quadrilateral element; Area=$\frac{9}{16}$

Eigen value computation on symmetric matrix stored in lower triangular
form. No stabilization. alpha=0.0
Lower triangular matrix:
```
    4.5679012346e−01
    2.5925925926e−01    6.9444444444e−01
   −4.5679012346e−01   −2.5925925926e−01    4.5679012346e−01
   −2.5925925926e−01   −6.9444444444e−01    2.5925925926e−01    6.9444444444e−01
```
Eigenvalues:
```
   −5.6255300012e−16   −4.8069663422e−17    5.8084783039e−01    1.7216213054e+00
```
Eigenvectors:
```
    7.0197435793e−01    8.5041171229e−02    5.9511652441e−01   −3.8188522147e−01
   −8.5041171229e−02    7.0197435793e−01   −3.8188522147e−01   −5.9511652441e−01
    7.0197435793e−01    8.5041171229e−02   −5.9511652441e−01    3.8188522147e−01
   −8.5041171229e−02    7.0197435793e−01    3.8188522147e−01    5.9511652441e−01
```

Eigen value computation on symmetric matrix stored in lower triangular
form. With stabilization. alpha=.0001
Lower triangular matrix:
```
    4.5683456790e−01
    2.5921481481e−01    6.9448888889e−01
   −4.5674567901e−01   −2.5930370370e−01    4.5683456790e−01
   −2.5930370370e−01   −6.9440000000e−01    2.5921481481e−01    6.9448888889e−01
```
Eigenvalues:
```
   −1.3766994923e−16    1.7777777778e−04    5.8084783039e−01    1.7216213054e+00
```
Eigenvectors:
```
   −5.0000000000e−01    5.0000000000e−01    5.9511652441e−01   −3.8188522147e−01
   −5.0000000000e−01   −5.0000000000e−01   −3.8188522147e−01   −5.9511652441e−01
   −5.0000000000e−01    5.0000000000e−01   −5.9511652441e−01    3.8188522147e−01
   −5.0000000000e−01   −5.0000000000e−01    3.8188522147e−01    5.9511652441e−01
```

8   Introduction to Magnetization of Materials

Two basic types of magnetic materials are introduced and reviewed: diamagnetic, and paramagnetic. These materials do not manifest magnetic hysteresis and are representative of magnetic materials *Alegra 2D* can currently model. This information is commonly known and available from various sources and websites [2, 5, 6]. Hysteresis and hence ferromagnetic materials are not discussed.

Magnetic fields arise due to charges in motion and are fundamentally a physical phenomenon induced and understood at the atomic scale through quantum mechanics – well beyond the discussion here. However, for continuum modeling purposes, it is helpful to have a basic understanding of the physics of magnetism. This then becomes helpful in matching mathematical models to specific materials. At the atomic scale, electrons spinning about nuclei can conceptually be thought of as tiny currents which induce magnetic dipoles. The aggregation of these tiny magnetic dipoles in the bulk material and their response to applied magnetic fields is how magnetic materials are classified.

Diagmagnetic materials   When a material consists of atoms/molecules with no unpaired electrons, the net magnetic field in the bulk material is zero. However, when an external field is applied to the material, in addition to the Coulomb force of attraction towards the nuclei, a Lorenz force acts on electrons either speeding them up or slowing them down. This change in angular momentum of the electron is in the opposite direction of the applied magnetic field; hence there is a small induced magnetic moment in the bulk in the opposite direction of the applied field [2, 6]. This accounts for why these materials have a relative permeability slightly less than 1. When the applied field is removed, the Lorenz force is removed and the material returns to its original state with no net magnetization. Diamagnetic materials are not attracted to an applied field. Note that the diamagnetic response described here is very small compared to a ferromagnetic response. A constitutive model such as that currently in *Alegra2D* may be relevant to diamagnetic materials.

Paramagnetic materials   Paramagnetic materials typically have unpaired electrons which in aggregate at the continuum scale, have magnetic dipoles which tend to cancel each other out because of random atom/molecule orientations. Thus, in the bulk, paramagnetic materials have no net magnetic field. In contrast with diamagnetic materials, unpaired electrons are free to orient themselves in response to an applied field. The tiny magnetic dipoles experience a torque which tends to align them with the applied field [5]. At a temperature of absolute zero, the extent to which dipoles align themselves is at a maximum. At room temperature, thermal agitation tends to prevent the material from completely aligning with the applied field. Whatever the case may be, the net alignment enhances/amplifies the applied field and the material is magnetized and hence paramagnetic materials have a relative permeability slightly greater than 1. The paramagnetic response to an applied field is relatively small compared with a ferromagnetic response. Paramagnetic materials are attracted to an applied field and do not exhibit hysteresis. When the applied field is removed, magnetization due to parallel alignment of dipoles is removed and the material returns to a state

of zero magnetization. A constitutive model such as that currently in *Alegra2D* may be relevant to paramagnetic materials.

9 Summary

A quadrilateral element for 2*D* magnetics calculations was presented. This element uses single point quadrature and is applicable to linear and nonlinear implicit application computer codes where the curl operator is needed. Because the element uses reduced order quadrature it must be stabilized. The report presented the stabilization which is easily incorporated into an implementation; a sample implementation was provided. Some simple but representative calculations on an element were done illustrating the element null space and how the stabilization perfectly eliminates the spurious mode while leaving the remaining modes undisturbed. For implicit transient magnetic calculations, or magnetostatic calculations involving nonlinear paramagnetic or diamagnetic materials, the linearized element stiffness matrix was developed an presented.

References

[1] D. P. Flanagan and T. Belytschko. A uniform strain hexahedron and quadrilateral with orthogonal hourglass control. *International Journal for Numerical Methods in Engineering*, 17(5):679–706, 1981.

[2] David J. Griffiths. *Introduction to Electrodynamics*. Prentice Hall, 2nd edition, 1989.

[3] Thomas A. Hail and Allen C. Robinson. Two-dimensional finite-element magnetohydrodynamic equations in alegra, part ii: Vector potential formulation. Sandia Report SAND2005-draft, Sandia National Laboratories, Albuquerque, NM, January 2005.

[4] Wing Kam Liu and Ted Belytschko. Efficient linear and nonlinear heat conduction with a quadrilateral element. *International Journal for Numerical Methods in Engineering*, 20(5):931–948, may 1984.

[5] Edward Della Torre. *Magnetic Hysteresis*. IEEE Press, 1999.

[6] Wikipedia. Magnetism — Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Magnetism, 2012. [Online; accessed 23-May-2012].

A Fortran coding example for mean curl on quadrilateral

```fortran
subroutine mean_curl_phi(xy,mean_curl)
    implicit none
    integer,parameter::npe=4, dimension=2
    real*8,intent(in)::xy(dimension,npe)
    real*8,intent(inout)::mean_curl(npe,dimension)
    real*8,dimension(0:3)::x,y
    real*8::area,a_inv
    x(0:3)=xy(1,1:4)
    y(0:3)=xy(2,1:4)
    ! area of quad
    area=0.5 * ( (x(3)-x(1))*(y(0)-y(2))+(x(0)-x(2))*(y(1)-y(3)) )
    a_inv = 1.0/area
    mean_curl(1,1)=0.5 * a_inv * (x(3)-x(1))
    mean_curl(1,2)=0.5 * a_inv * (y(3)-y(1))
    mean_curl(2,1)=0.5 * a_inv * (x(0)-x(2))
    mean_curl(2,2)=0.5 * a_inv * (y(0)-y(2))
    mean_curl(3,1)=0.5 * a_inv * (x(1)-x(3))
    mean_curl(3,2)=0.5 * a_inv * (y(1)-y(3))
    mean_curl(4,1)=0.5 * a_inv * (x(2)-x(0))
    mean_curl(4,2)=0.5 * a_inv * (y(2)-y(0))
end subroutine mean_curl_phi


subroutine element_stiffness(xy,eK,mat,alpha)
!    Input:
!    xy: element coordinates
!    mat: scalar coefficient associated with material
!    alpha: scalar hourglass constant used for stabilization
!    Output:
!        * Element stiffness 'eK' is computed
!        * 1) symmetric 4x4 matrix
!        * 2) stored in lower triangular form
!        * 3) Note storage
!            K11
!            K21 K22
!            K31 K32 K33
!            K41 K42 K43 K44
    implicit none
    integer,parameter::npe=4, dimension=2, length=10
    real*8,intent(in)::xy(dimension,npe)
    real*8,intent(in)::mat,alpha
    real*8,intent(inout)::eK(length)
    ! mc is mean_curl
    real*8,dimension(npe,dimension)::mc
    real*8,dimension(0:3)::x,y
    real*8,parameter,dimension(0:3)::hg=(/1.0,-1.0,1.0,-1.0/)
```

```fortran
      real *8:: area ,g
      x(0:3)= xy(1 ,1:4)
      y(0:3)= xy(2 ,1:4)

      ! area of quad
      area =0.5  *  (  (x(3)−x(1))*(y(0)−y(2))+(x(0)−x(2))*(y(1)−y(3))  )

      ! mean curl of shape functions
      call mean_curl_phi(xy,mc)

      ! form element stiffness by tensor product
      ! column 1
      eK(1) = mat * area * ( mc(1,1) * mc(1,1) + mc(1,2) * mc(1,2) )
      eK(2) = mat * area * ( mc(2,1) * mc(1,1) + mc(2,2) * mc(1,2) )
      eK(3) = mat * area * ( mc(3,1) * mc(1,1) + mc(3,2) * mc(1,2) )
      eK(4) = mat * area * ( mc(4,1) * mc(1,1) + mc(4,2) * mc(1,2) )

      ! column 2
      eK(5) = mat * area * ( mc(2,1) * mc(2,1) + mc(2,2) * mc(2,2) )
      eK(6) = mat * area * ( mc(3,1) * mc(2,1) + mc(3,2) * mc(2,2) )
      eK(7) = mat * area * ( mc(4,1) * mc(2,1) + mc(4,2) * mc(2,2) )

      ! column 3
      eK(8) = mat * area * ( mc(3,1) * mc(3,1) + mc(3,2) * mc(3,2) )
      eK(9) = mat * area * ( mc(4,1) * mc(3,1) + mc(4,2) * mc(3,2) )

      ! column 4
      eK(10) = mat * area * ( mc(4,1) * mc(4,1) + mc(4,2) * mc(4,2) )

      ! add hourglass stabilization
      g = alpha *(mat /4.0/ area )
      eK(1)  = eK(1)  + g * hg(0)*hg(0)
      eK(2)  = eK(2)  + g * hg(1)*hg(0)
      eK(3)  = eK(3)  + g * hg(2)*hg(0)
      eK(4)  = eK(4)  + g * hg(3)*hg(0)
      eK(5)  = eK(5)  + g * hg(1)*hg(1)
      eK(6)  = eK(6)  + g * hg(2)*hg(1)
      eK(7)  = eK(7)  + g * hg(3)*hg(1)
      eK(8)  = eK(8)  + g * hg(2)*hg(2)
      eK(9)  = eK(9)  + g * hg(3)*hg(2)
      eK(10) = eK(10) + g * hg(3)*hg(3)

end subroutine element_stiffness
```

DISTRIBUTION:

1   MS 0899   Technical Library, 9536

Sandia National Laboratories