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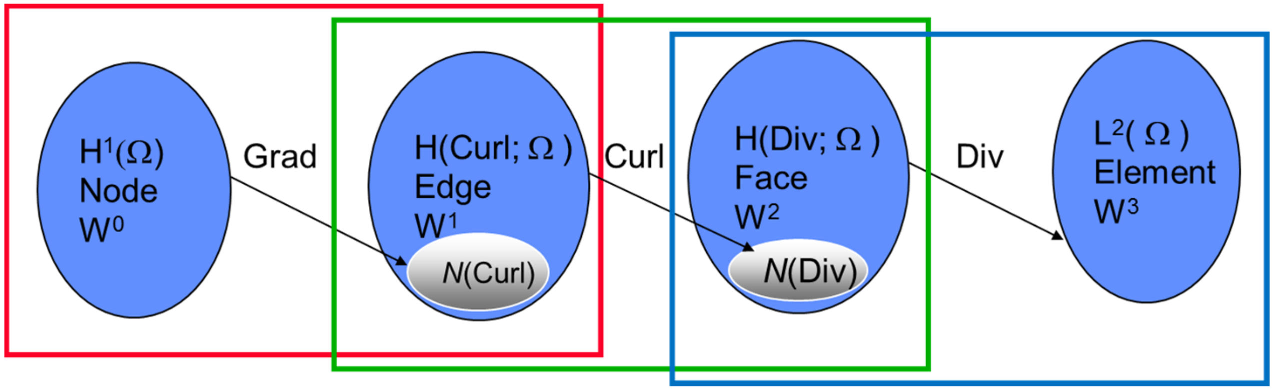


# Lagrangian/Eulerian Multiphysics Modeling and DeRham Complex Based Algorithms


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High Resolution Mathematical and Numerical Analysis of Involution-Constrained PDEs

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# Outline

- Lagrangian/Eulerian Numerical Methods
- DeRham Tour
  - Inverse Deformation Gradient
  - Magnetic Flux Density
  - Volume Remapping
  - A possible cross-cutting algorithm
- Conclusion

# Arbitrary Lagrangian/Eulerian (ALE)

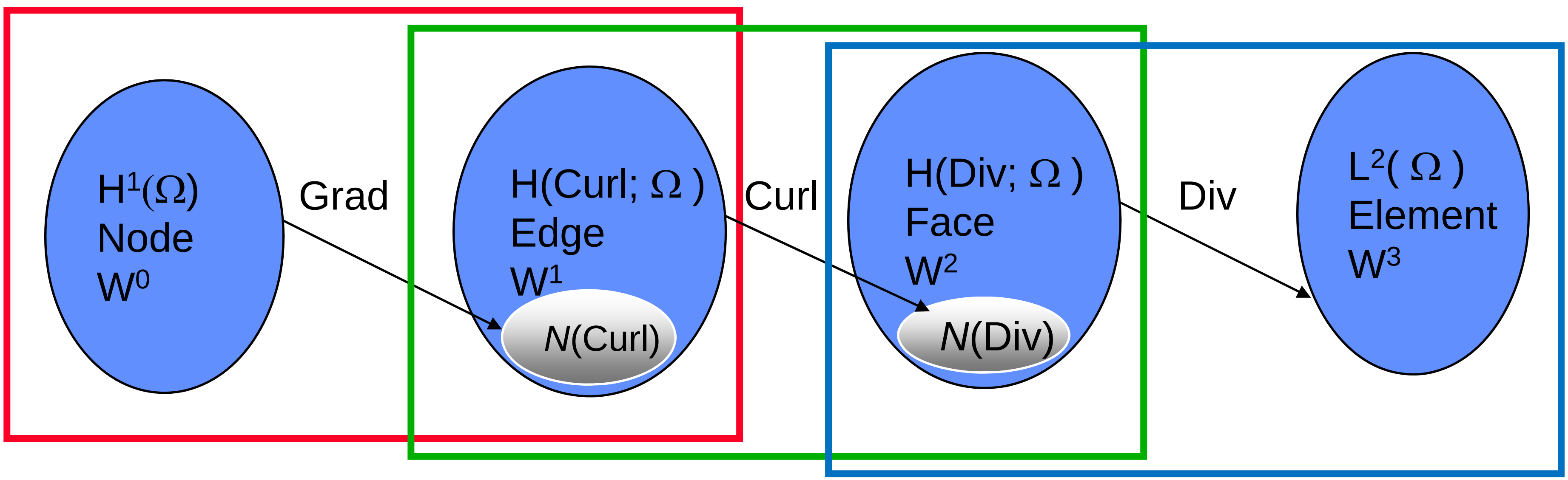
- **Lagrangian:**
  - Mesh moves with material points.
  - Mesh-quality may deteriorate over time
- **REMESH**
  - Mesh-quality is adjusted to improve solution-quality or robustness.
  - **Eulerian** sets new mesh to original location
- **REMAP**
  - Algorithm transfers dependent variables to the new mesh.

# What happens with Involution Constraints and ALE?

- **Lagrangian:**
  - The kinematic complexity is simplified due to embedding in the Lagrangian frame.
  - Use of mimetic operators keeps the solution in the right space.
- **Remesh:**
  - Nothing special
- **Remap**:
  - Algorithm looks like a “constrained transport” algorithm in some way.
  - The algorithm of necessity is un-split.

# Geometric Structure and Numerical Methods

- The structure of the equations is related to their geometric origins.
- This geometry can reappear in effective numerical methods.
- The deRham structure shown below is used to discuss issues of “compatible discretizations.”
- These are related to 0-forms, 1-forms, 2-forms and 3-forms.
- Transport theorems are associated with the kinematics of such mathematical ideas.
- Presentation is “color coded”



# Circulation Transport Theorem

$$\frac{d}{dt}\int_{\phi_t(U)} A_1\,dx + A_2\,dy + A_3\,dz = \frac{d}{dt}\int_{\phi_t(U)} A_k\,dx_k$$

$$= \int_{\phi_t(U)} \dot{A}_k\,dx_k + A_k\frac{\partial \mathrm{v}_k}{\partial x_l}dx_l = \int_{\phi_t(U)} \dot{A}_k\,dx_k + A_l\frac{\partial \mathrm{v}_l}{\partial x_k}dx_k$$

$$= \int_{\phi_t(U)} \left(\frac{\partial A_k}{\partial t} + \mathrm{v}_l\frac{\partial A_k}{\partial x_l} + A_l\frac{\partial \mathrm{v}_l}{\partial x_k}\right)dx_k$$

$$= \int_{\phi_t(U)} (\mathbf{A}_t + \nabla(\mathbf{A}\bullet\mathbf{v}) - \mathbf{v}\times(\nabla\times\mathbf{A}))\bullet d\mathbf{x}$$

# Surface Flux Transport Theorem

$$\frac{d}{dt}\int_{\phi_t(U)} B_1\, dy\, dz + B_2\, dz\, dx + B_3\, dx\, dy = \frac{d}{dt}\int_{\phi_t(U)} B_k\, da_k$$

$$= \int_{\phi_t(U)} \dot{B}_1\, dy\, dz + B_1\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right)dz$$

$$+B_1 dy\left(\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz\right) + \ldots$$

$$= \int_{\phi_t(U)} \left(\dot{B}_i + B_i\frac{\partial \mathrm{v}_k}{\partial x_k} - B_k\frac{\partial \mathrm{v}_i}{\partial x_k}\right)da_i$$

$$= \int_{\phi_t(U)} (\mathbf{B}_t + \nabla \times (\mathbf{B} \times \mathbf{v}) + \mathbf{v}\,\mathrm{div}\mathbf{B})_i\, da_i$$

# Volume Transport

$$\frac{d}{dt}\int_{\phi_t(U)} \rho\, dv = \frac{d}{dt}\int_U \rho(\phi(X,t),t)J(X,t)\, dV =$$

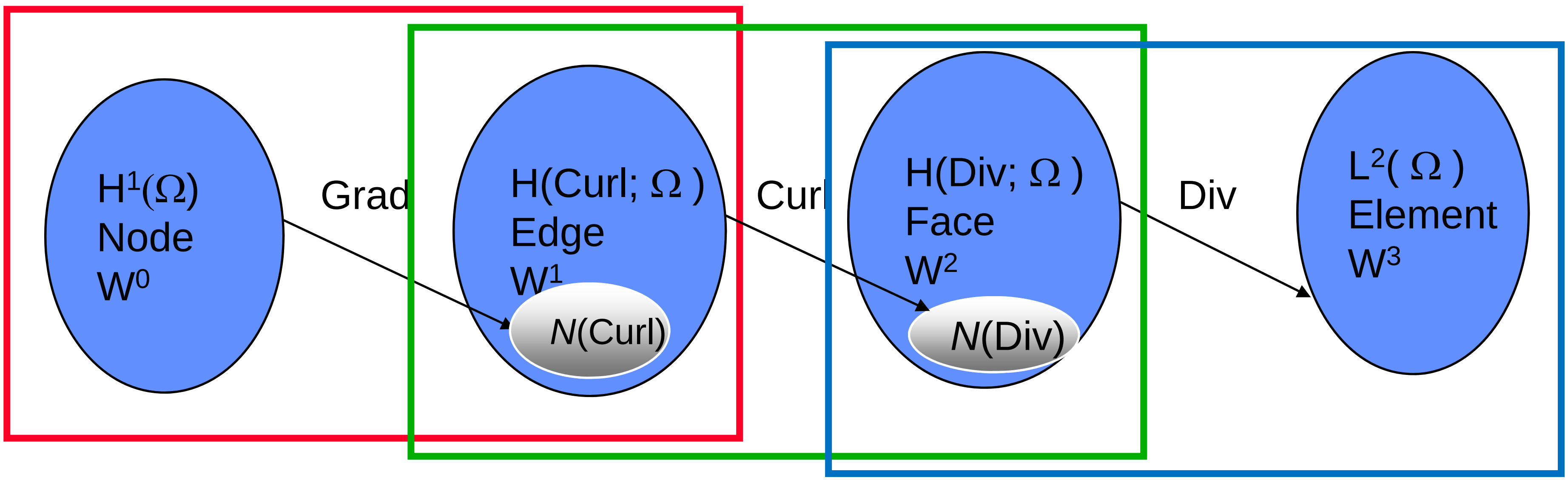
$$= \int_U \dot{\rho}(\phi(X,t),t)J(X,t) + \rho(\phi(X,t),t)\dot{J}(X,t)\, dV$$

$$= \int_U (\rho_t + \mathbf{v}\bullet\nabla\rho)J(X,t) + \rho(\phi(X,t)(\mathrm{div}\ \mathbf{v})J(X,t)\, dV$$

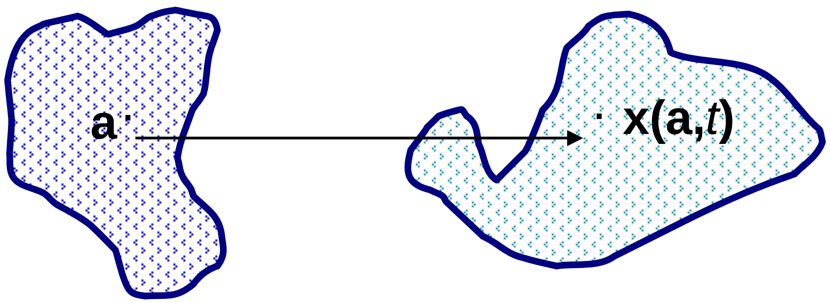
$$= \int_{\phi_t(U)} (\dot{\rho} + \rho\,\mathrm{div}\ \mathbf{v})\, dv = \int_{\phi_t(U)} (\rho_t + \mathrm{div}\ (\rho\mathbf{v}))\, dv$$

$$= \int_{\phi_t(U)} (\rho_t + \mathrm{div}\ (\rho\mathbf{v}))\, dv$$

# Solid Kinematics



# Solid Kinematics




**(Reference) Material Coordinates**

x(a,*t*)

**(Current) Spatial Coordinates**

Deformation gradient and inverse:

$$\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{a}$$

$$\mathbf{G} = \mathbf{F}^{-1} = \partial \mathbf{a} / \partial \mathbf{x}$$

***Polar Decomposition*: F = VR**

Symmetric Positive Definite
(Stretch) Tensor

Proper Orthogonal
(Rotation) Tensor

# Remap

- Some material models require that the kinematic description (i.e. F) be available. The rotation tensor in particular is needed.
- Any method for tracking F on a discrete grid may fail eventually.
  - Det(F)>0
  - Positive definiteness of the stretch, V, can be lost.
  - R proper orthogonal: $RR^T = I$, Det(R)>0.
  - Rows of the inverse deformation tensor $G=F^{-1}$ should be gradients.
- These constraints may not hold due to truncation errors in the remap step and finite accuracy discretizations.
- What is the best approach?
  - "fixes" will be required.
  - Storage, accuracy and speed should be considered.

# Possible Solutions

- Use an integration scheme to update V and R in the Lagrangian step using the rate-of-deformation tensor.
  - Conservatively remap components of both V and R (VR)
  - Conservatively remap components of V and quaternion parameters representing R (QVR)
- We have investigated a constrained transport remap to stay in a curl free space (DG)
- Apply appropriate fixes or projections where possible and necessary.

# The stretch can fail to be positive definite after remap (VR/QVR)

Limiting minimum and maximum stretches enables robustness.

Spectral Decomposition

$$\mathbf{V}\mathbf{Q} = \mathbf{Q}\Lambda$$

Eigenvectors

Eigenvalues

$$\hat{\lambda}_k = \min(\max(\lambda_k, \lambda_s), 1/\lambda_s)$$

$$\hat{\mathbf{V}} = \mathbf{Q}\hat{\Lambda}\mathbf{Q}^T$$

# Project R to rotation after remap

2D (VR)

$$\begin{aligned}\hat{R}_{11} &= (R_{11} + R_{22})/a \\ \hat{R}_{21} &= (R_{21} - R_{12})/a \\ \hat{R}_{12} &= (R_{12} - R_{21})/a \\ \hat{R}_{22} &= (R_{11} + R_{22})/a\end{aligned}$$

$$a = \sqrt{(R_{11} + R_{22})^2 + (R_{21} - R_{12})^2}$$

3D (VR)

$$\mathbf{R}^0 = \sqrt{\frac{3}{tr(\mathbf{R}^T\mathbf{R})}}\mathbf{R}.$$

$$\mathbf{R}^{m+1} = \frac{1}{2}\mathbf{R}^m[3\mathbf{I} - (\mathbf{R}^m)^T\mathbf{R}^m]$$

QVR

$$q = q_r/\sqrt{q_r\bar{q}_r}.$$

# Comparison of 2D ALE Rotation Algorithms for Two Test Problems

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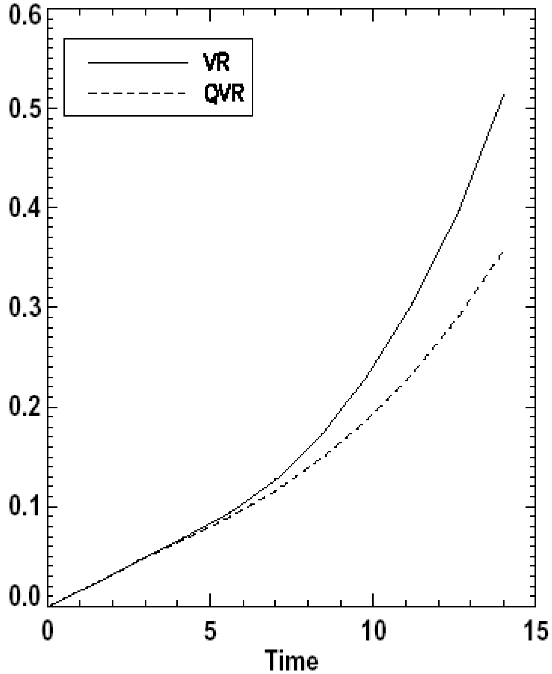
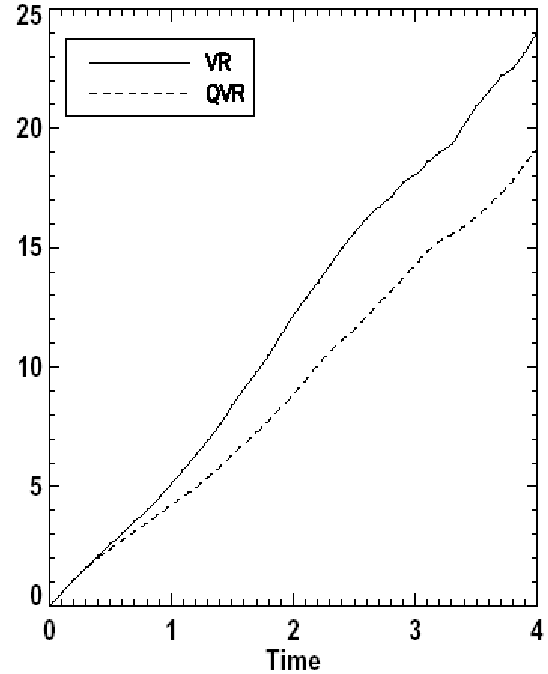
Exponential Vortex

ABC Rotate

$$\mathbf{v}_\theta = \frac{\Gamma}{2\pi r}(1 - e^{-r^2/2})$$




$$\mathbf{v}_\theta = \omega_0 r \;\; 0 < r < a$$

$$\mathbf{v}_\theta = \frac{\omega_0 a^2}{r} \;\; a < r < b$$

$$\mathbf{v}_\theta = \frac{\omega_0 a^2}{r}\left(\frac{c^2 - r^2}{c^2 - b^2}\right) \;\; b < r < c$$

$$\mathbf{v}_\theta = 0 \;\; c < r$$

Relative error growth for test problems comparing quaternion with exponential map algorithm (QVR) versus rotation tensor with Cayley transformation (VR)

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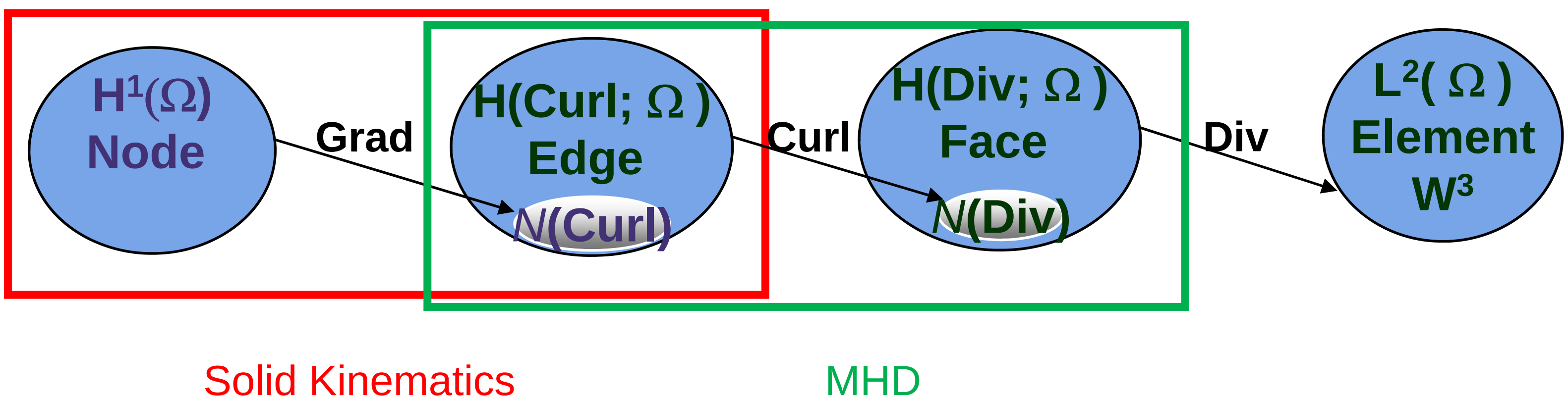
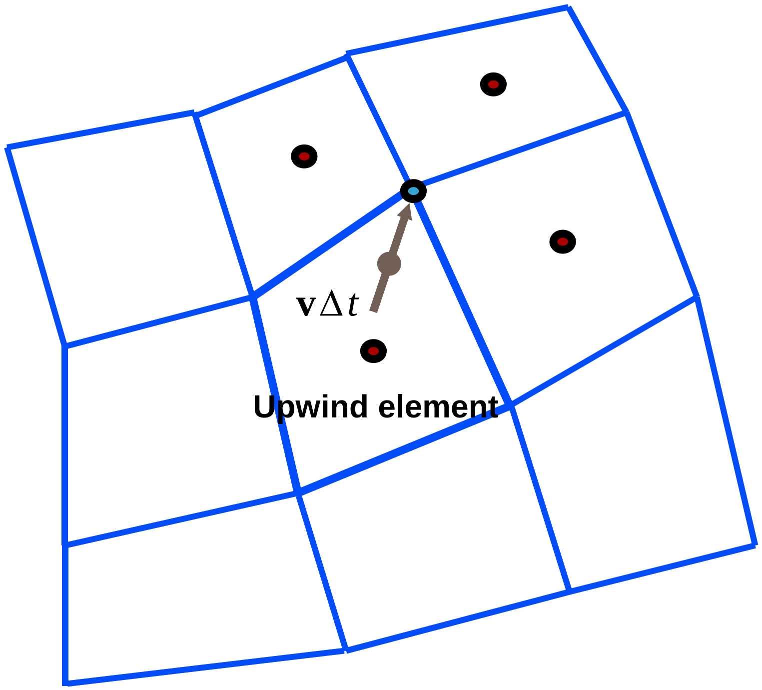
# Curl Free Constrained  Transport (DG)

- Is there something more satisfying?
- Representation of G on edges allows for a discrete curl-free inverse deformation gradient.
- Remap algorithm should preserve this property.
- Constrained transport (CT) approach pioneered by Evans and Hawley for div free MHD algorithm on Cartesian grid is the prototype algorithm.



# Curl Free Remap Algorithm




Rows guaranteed to be curl free. ☺

No control on det(G). ☹

Speed ☹

- Edge element representation

$$\mathrm{g}(\xi_1,\xi_2,\xi_3) = \sum_{i\neq j\neq k,\alpha,\beta} \Gamma_{ij}^{\alpha\beta}(\xi_k)\hat{W}_{ij}^{\alpha\beta}$$

- Use patch recovered nodal values of G to compute trial edge element gradient coefficients along each edge.

$$\Gamma_{ij}^{\alpha\beta}(\xi_k) = \bar{\Gamma}_{ij}^{\alpha\beta} + s_{ij}^{\alpha\beta}\xi_k$$

- Limit slopes along each edge (minmod,harmonic)
- Compute the node circulation contributions in the upwind element by a midpoint integration rule at the center of the node motion vector.

$$\int_{\Gamma} \mathrm{g}\cdot ds \approx \sum_{i\neq j\neq k,\alpha,\beta} \Gamma_{ij}^{\alpha\beta}(\hat{\xi}_k)(1+\alpha\hat{\xi}_i)(1+\beta\hat{\xi}_j)\delta\xi_k/8$$

- Take gradient and add to edge element circulations.

# Solid Kinematics Remap

- There are significant benefits for quaternion rotation (LQVR,QVR) representation with volumetric remap.
- Stretch tensor reset algorithm based on eigenvalue decomposition has been shown to provide robustness.
- Inverse deformation gradient modeling with curl free remap required continued investigation. SAND2009-5154
- BIG question #1: How to control det(G)?
- BIG question #2: How to program the CT algorithms <u>efficiently</u>? In particular one needs to find the upwind element.
- Research Question: The det(G) constraint essentially links a CT type algorithm across 2 or 3 coordinates. Is there a better (perhaps more coordinate free) way to think about the problem?

# One possible approach to solving the det(G)>0 problem

- Kamm, Love, Ridzal, Young, Robinson have investigated whether optimization based remap ideas might help.
- Solve global optimization problem for nodal increments using the standard CT algorithm increments as the target.

$$\min_u f(u) \quad \text{subject to} \quad g(u) = 0 \quad \text{and} \quad h(u) > 0 .$$

$$f(u) := \frac{1}{2}\sum_i (u_i - \hat{u}_i)^2 \qquad\qquad h_j(u) := \det{}_j(u) - \varepsilon > 0 \quad \text{with} \quad \varepsilon := \min_{k \in \mathcal{K}} \{\det{}_k(u^L)\}$$

- Solve using slack variable formulation

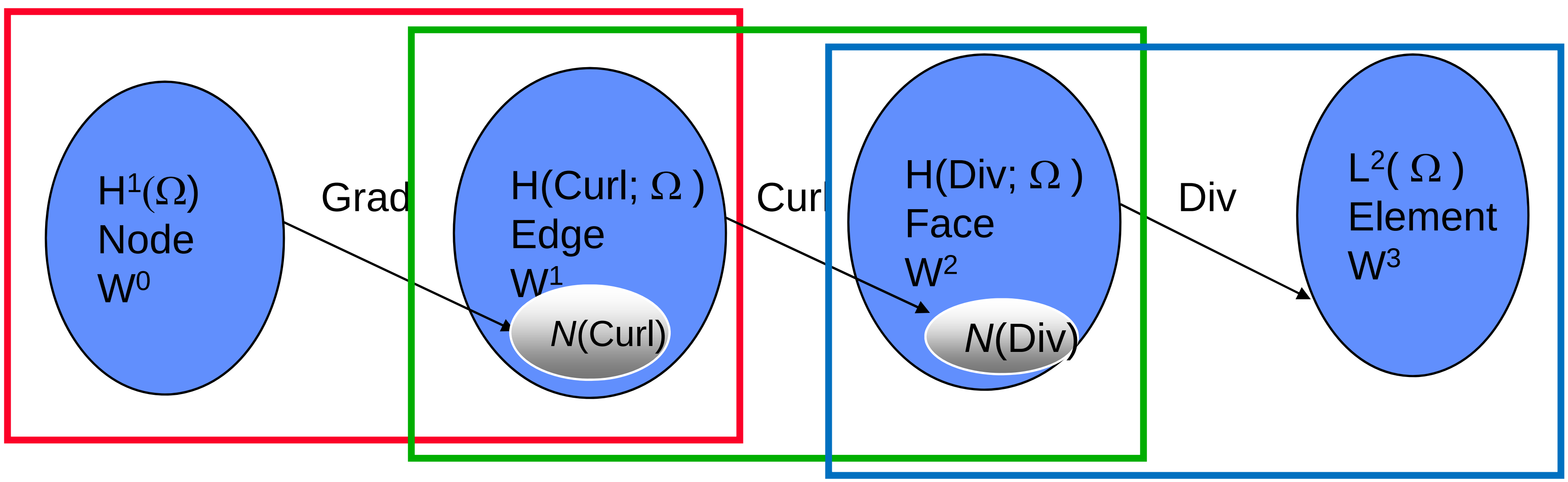
$$\min_u f(u) \quad \text{subject to} \quad g(u) = 0,\, h(u) - s = 0 \quad \text{and} \quad s - \varepsilon > 0$$

- Research report in progress.
- Key idea: optimization might be able to help with remap.

# Eulerian Frame for kinematics

- Caltech group has had success with Eulerian frame equations for solid kinematics.
- The G equation (circulation transport theorem for three components of inverse deformation gradient) must contain a term related to preserving consistency with mass conservation.
- Phil Barton
  - Caltech and now at AWE
  - Reports success with both F and G equations. (Personal communication at Multimat 2013 (with permission)).
  - Did not use the additional diffusion term of Miller and Colella.
- See also Hill, Pullin, Ortiz, Meiron JCP 229 (2010) and Miller and Colella, JCP 167 (2001).
- Is there something to be learned from Eulerian frame success for ALE algorithms? Are there weaknesses about Eulerian frame that are not clear?

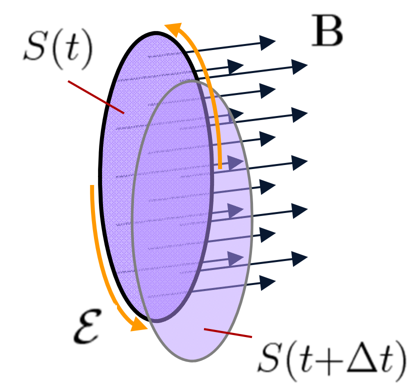
# Magnetohydrodynamics



# Faraday's Law (Natural operator splitting)

A straightforward **B**-field update is possible using Faraday's law.

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \mathcal{E} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$




Integrate over time-dependent surface $S(t)$ , apply Stokes theorem, and discretize in time:

$$\frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot d\mathbf{a} + \oint_{\partial S(t)} \mathcal{E}\cdot d\mathbf{x} = 0$$

$$\frac{1}{\Delta t}\int_{S(t+\Delta t)} (\mathbf{B}^{n+1} - \tilde{\mathbf{B}}^{n+1})\cdot d\mathbf{a}^{n+1} + \oint_{\partial S(t+\Delta t)} \mathcal{E}^{n+1}\cdot d\mathbf{x}^{n+1}$$

$$+ \frac{1}{\Delta t}\left[\int_{S(t+\Delta t)} \tilde{\mathbf{B}}^{n+1}\cdot d\mathbf{a}^{n+1} - \int_{S(t)} \mathbf{B}^{n}\cdot d\mathbf{a}^{n}\right] = 0$$

Zero **for ideal MHD** by frozen-in flux theorem:

$$\boxed{\frac{d}{dt}\int_{S_t} \mathbf{B}\cdot d\mathbf{a} = \int_{S_t} \overset{*}{\mathbf{B}}\cdot d\mathbf{a} = 0}$$

Terms in red are zero for ideal MHD so nothing needs to be done if fluxes are degrees of freedom.

# Solve magnetic diffusion using edge/face elements which preserve discrete divergence free property

$\Omega$ = a single conducting region in $\Re^3$.

weakly enforced

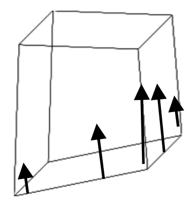
$$\nabla \times \mathbf{H} = \mathbf{J} \qquad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

Exact relationship

$$\nabla \bullet \mathbf{J} = 0 \qquad \nabla \bullet \mathbf{B} = 0$$

$$\mathbf{B} = \mu \mathbf{H} \qquad \mathbf{J} = \sigma \mathbf{E}$$

$$\text{boundary conditions } \begin{cases} \mathbf{E} \times \mathbf{n} = \mathbf{E}_b \times \mathbf{n} \text{ on } \Gamma_1 (\mathit{Dirichlet}), \\ \mathbf{H} \times \mathbf{n} = \mathbf{H}_b \times \mathbf{n} \text{ on } \Gamma_2 (\mathit{Neumann}) \end{cases}.$$



Edge element

$$\int \sigma \mathbf{E}^{n+1} \bullet \hat{\mathbf{E}} dV + \Delta t \int \frac{\text{curl}\, \mathbf{E}^{n+1} \bullet \text{curl}\, \hat{\mathbf{E}}}{\mu} dV = \int \frac{\mathbf{B}^n \bullet \mathbf{curl}\, \hat{\mathbf{E}}}{\mu} dV - \int \mathbf{H}_b \times \mathbf{n} \bullet \hat{\mathbf{E}} dA$$

B = magnetic flux density   E = electric field   H = magnetic field

$\mu$ = permeability   $\sigma$ = conductivity   J = current density

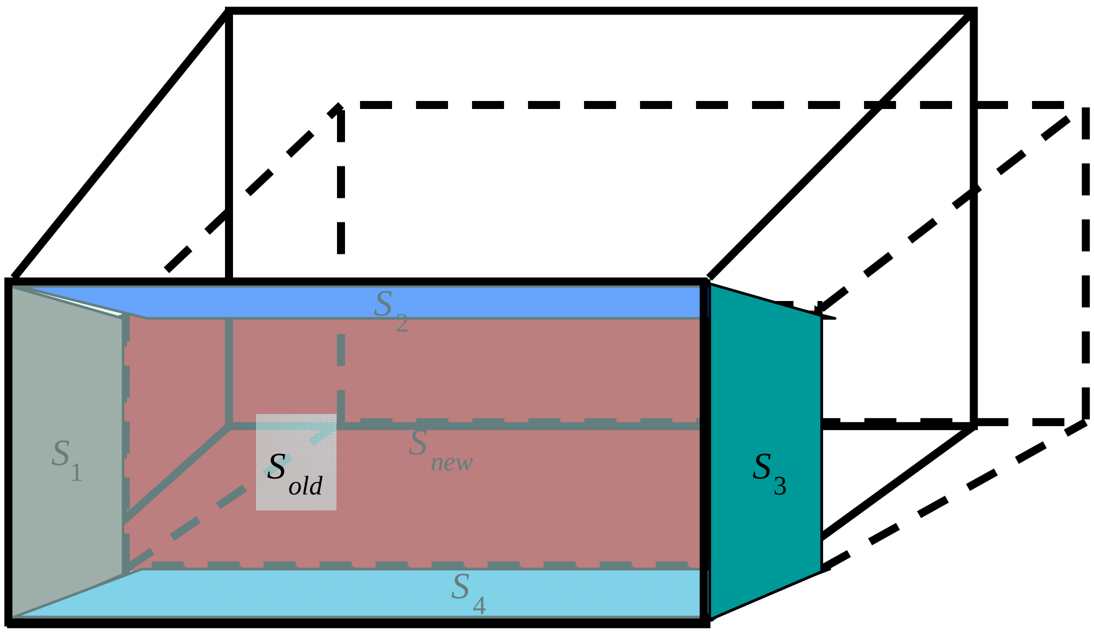
$\mu$ and $\sigma$ positive and finite everywhere in $\Omega$

# Magnetic Flux Density Remap

- The Lagrangian step maintains the discrete divergence free property via flux density updates given only in term of curls of edge centered variables.
- The remap should not destroy this property.
- Constrained transport is fundamentally unsplit.

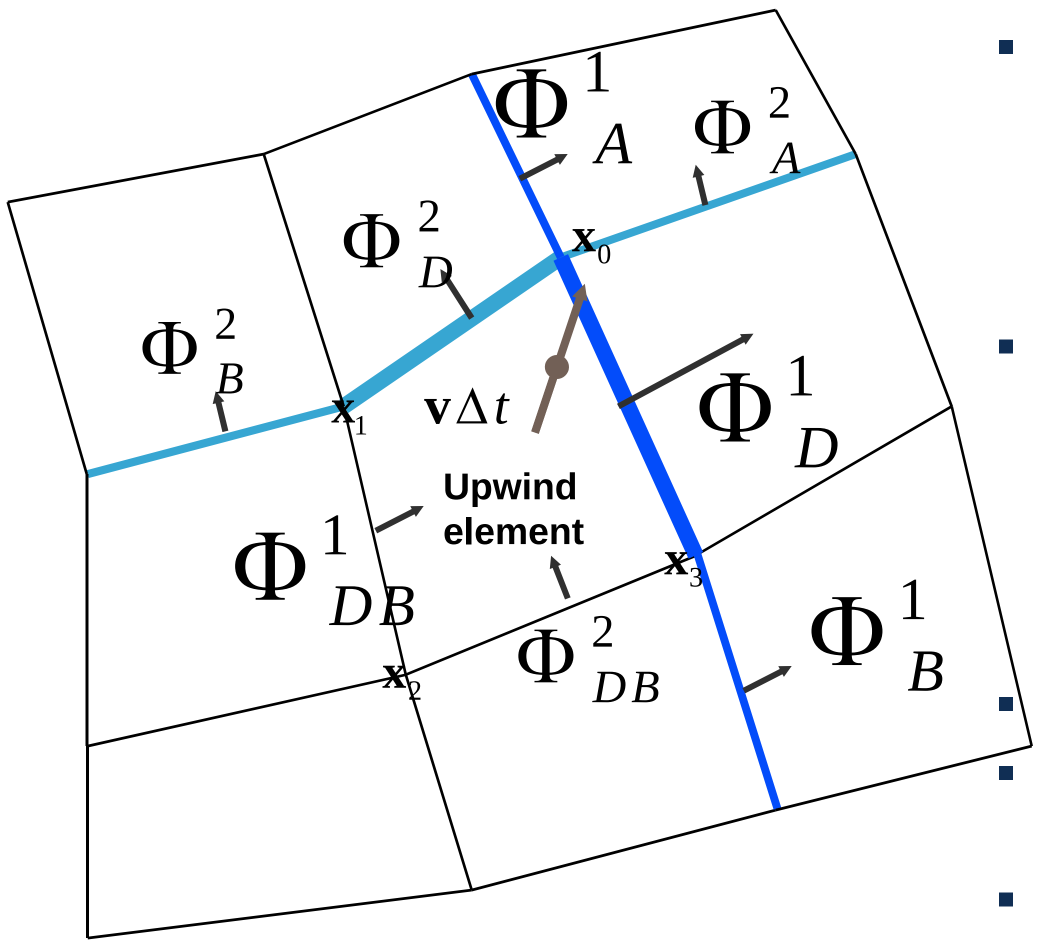
# Flux remap step

$$\int_S \mathbf{B} \bullet d\mathbf{a} = 0 \qquad \int_{S_{old}} \mathbf{B} \bullet d\mathbf{a} + \int_{S_{new}} \mathbf{B} \bullet d\mathbf{a} + \sum_{i=1}^{4} \int_{S_i} \mathbf{B} \bullet (\mathbf{v}_g \Delta t \times \mathbf{dl}) = 0$$




$$\int_{S_{old}} \mathbf{B} \bullet d\mathbf{a} + \int_{S_{new}} \mathbf{B} \bullet d\mathbf{a} + \sum_{i=1}^{4} \int_{S_i} \mathbf{dl} \bullet (\mathbf{B} \times \mathbf{v}_g \Delta t) = 0$$

# CT on unstructured quad and hex grids (CCT)




- Define the low order or donor method by integrating the total flux through the upwind characteristic of the total face element representation of the flux density.
- High order method constructs a modification to the flux so that it varies across the element face. Compute flux density gradients in the tangential direction using the blue and the red faces.
- All contributions are combined.
- Electric field updates are located on edges.
- Take curl to get updated fluxes.
- Requires tracking flux and circulation sign conventions.

# Face element representation

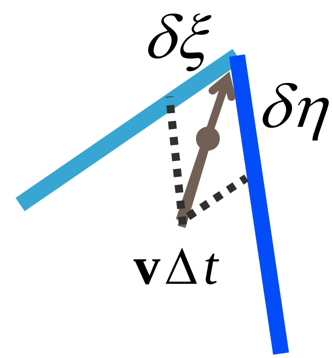
- Obtain representation of upwind element in terms of natural coordinates of an isoparametric element.

$$\mathbf{x} = (\mathbf{x}_0 + \xi(\mathbf{x}_1 - \mathbf{x}_0)) + \eta[(\mathbf{x}_3 + \xi(\mathbf{x}_2 - \mathbf{x}_3)) - (\mathbf{x}_0 + \xi(\mathbf{x}_1 - \mathbf{x}_0))]$$

$$\mathbf{B} = \sum_f \Phi_f \mathbf{F}_f = \frac{\Phi_D^1(\xi - 1)\frac{\partial \mathbf{x}}{\partial \xi}}{\frac{\partial \mathbf{x}}{\partial \xi} \times \frac{\partial \mathbf{x}}{\partial \eta}} + \frac{-\Phi_{DB}^1 \xi \frac{\partial \mathbf{x}}{\partial \xi}}{\frac{\partial \mathbf{x}}{\partial \xi} \times \frac{\partial \mathbf{x}}{\partial \eta}} + \frac{\Phi_D^2(\eta - 1)\frac{\partial \mathbf{x}}{\partial \eta}}{\frac{\partial \mathbf{x}}{\partial \xi} \times \frac{\partial \mathbf{x}}{\partial \eta}} + \frac{-\Phi_{DB}^2 \eta \frac{\partial \mathbf{x}}{\partial \eta}}{\frac{\partial \mathbf{x}}{\partial \xi} \times \frac{\partial \mathbf{x}}{\partial \eta}}$$

- **Integrate over flux surface.**

$$\int_{S_i} d\mathbf{x} \times \mathbf{B} \approx \delta\eta(\Phi_D^1 + \tfrac{\delta\xi}{2}(\Phi_{DB}^1 - \Phi_D^1)) - \delta\xi(\Phi_D^2 + \tfrac{\delta\eta}{2}(\Phi_{DB}^2 - \Phi_D^2))$$

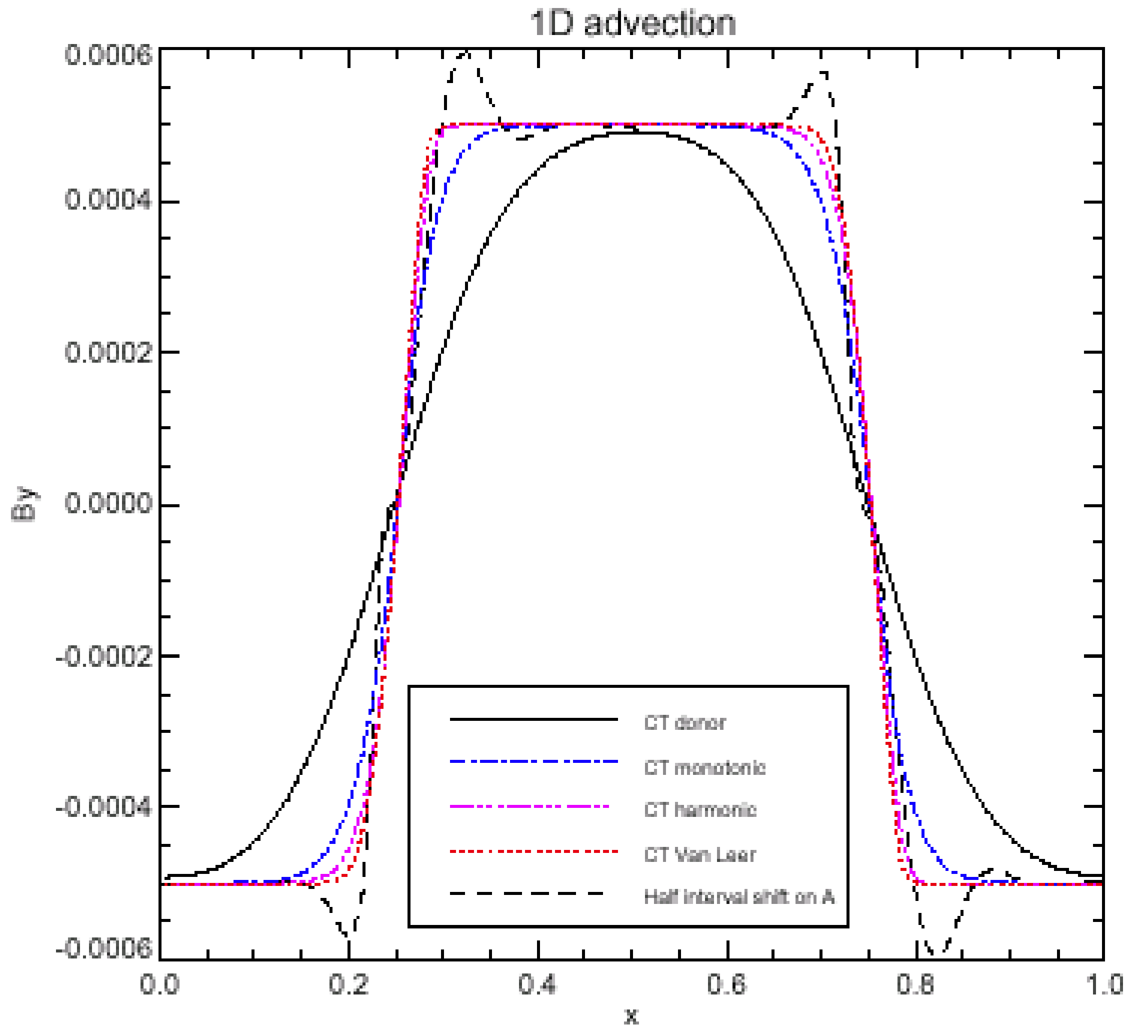



- **Normal gradient terms appear naturally.**

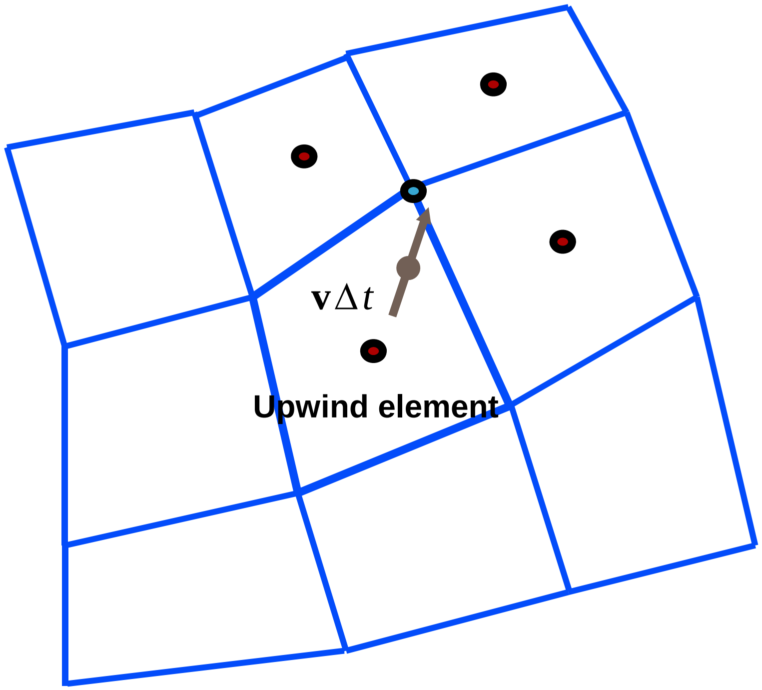
- **A cross face tangential gradient limiting is implemented**
- **Several limiters implemented (Van Leer, harmonic, minmod, donor)**

$$\widehat{\Phi}^1(\eta) = \Phi_D^1 + (A_D^1)^2 s^1(\tfrac{1}{2} - \eta) \qquad \widehat{\Phi}^2(\xi) = \Phi_D^2 + (A_D^2)^2 s^2(\tfrac{1}{2} - \xi)$$

# CT 1D advection

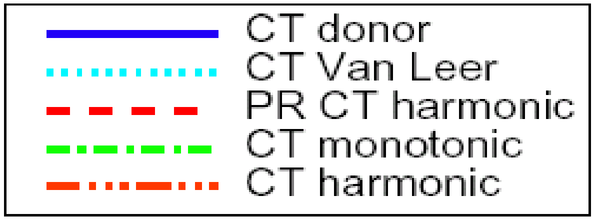


# Improved CCT Algorithm




- **Compute B at nodes from the face element representation at element centers. This must be second order accurate. Patch recovery (PR) suggested. Other means are possible.**
- **Compute trial cross face element flux coefficients on each face using these nodal B.**
- **Limit on each face to obtain cross face flux coefficients which contribute zero total flux.**
- **Compute the edge flux contributions in the upwind element by a midpoint integration rule at the center of the edge centered motion vector.**

- ``Arbitrary Lagrangian-Eulerian 3D Ideal MHD Algorithms,'' Int. Journal Numerical Methods in Fluids, 2011;65:1438-1450. (remap and deBar energy conservation discussed)
- Bochev and student have looked at optimization based reconstruction for flux based remap.
- The key thing to optimize is the magnetic energy loss.

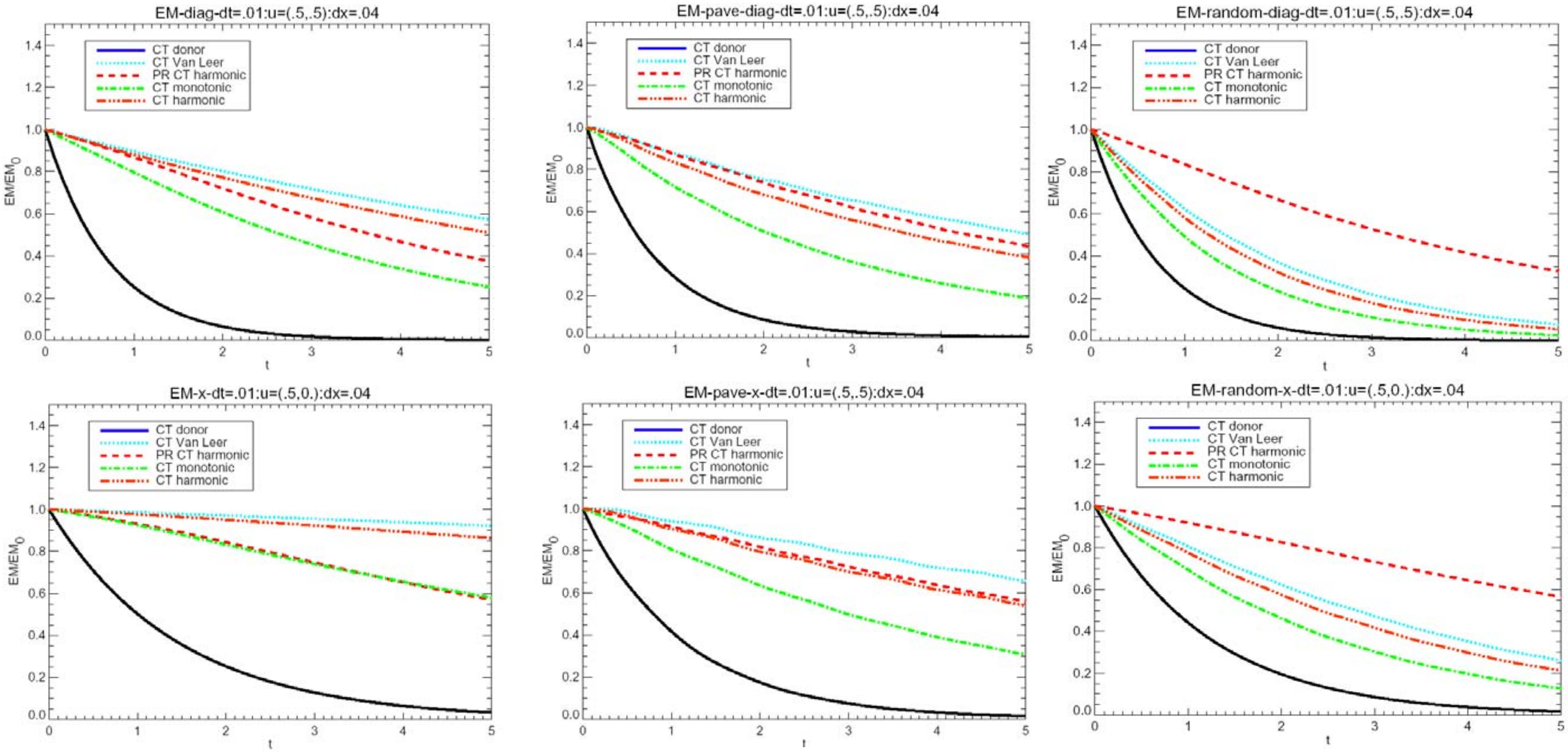



# Patch Recovery Based CCT

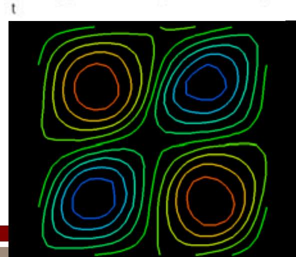
**Cartesian** | **Paved** | **Randomized**

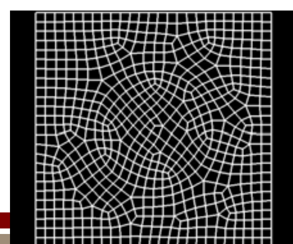
**diag**

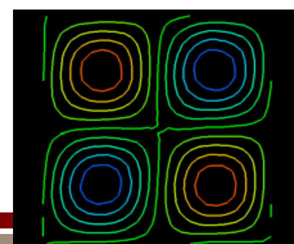
**x**




**Paved,diagonal, face based, harmonic**

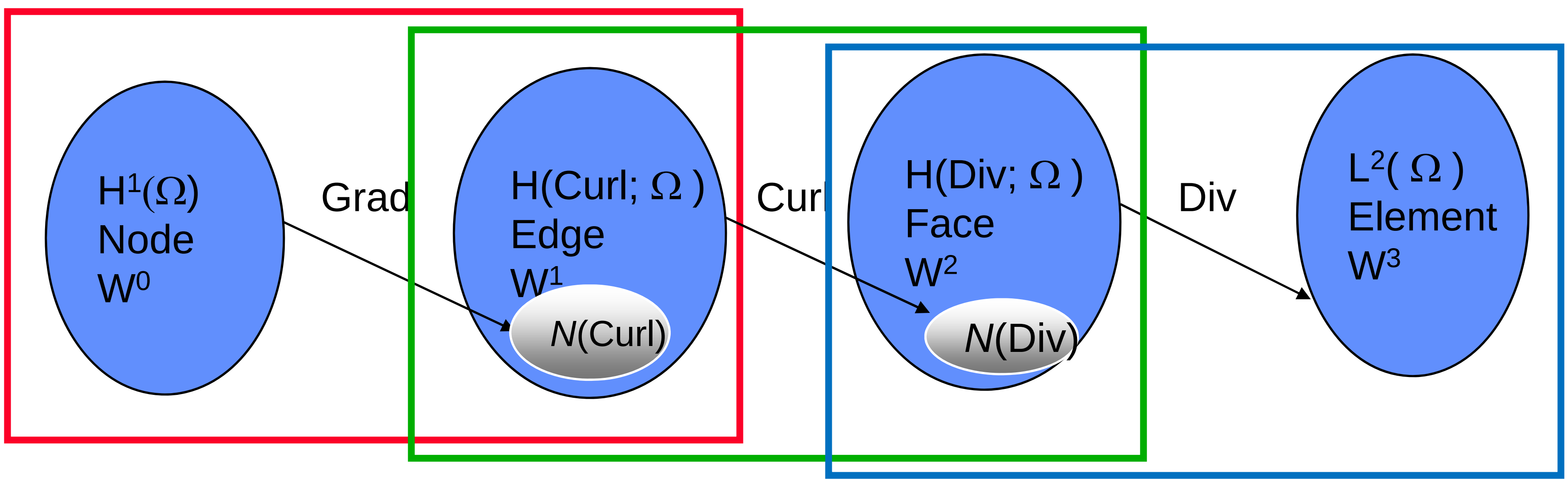






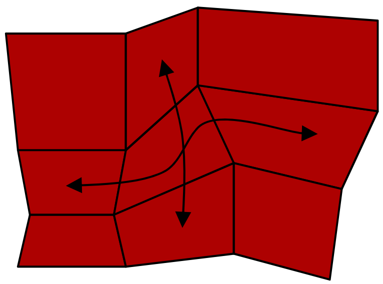
**Paved,diagonal, patch recovery, harmonic**

# Hydrodynamics



# Hydrodynamics

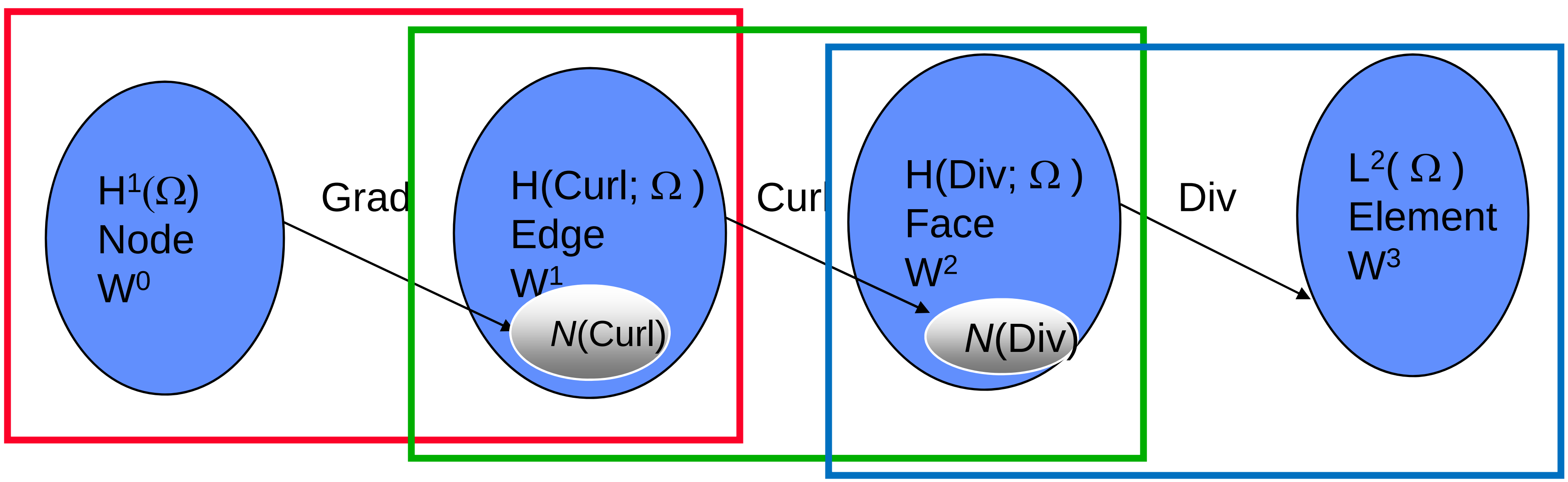
- Lagrangian Step
  - Mass is conserved in the Lagrangian frame.
  - Discrete Lagrangian continuity equation is trivial.



- Remap Step
  - Swept surfaces or overlap grids plus integration over reconstructed densities yield mass changes.
  - Remap algorithms associated with the blue box have been worked on for a long time.
  - Recent new algorithms tend to emphasize solving optimization problems to avoid excessive dissipation. See work by Shashkov and Bochev and their coworkers.

- My impression is that the blue box in the deRham diagram has received most of the research attention!

# Cross cutting algorithms



# Cross cutting algorithms

- Is it possible to build an ALE numerical method for the full Maxwell's equations coupled to mechanics that naturally transitions between the electro-quasi-static and magneto-quasi-static regimes, is reasonably efficient and would give a useful approximation to at least some low frequency electromagnetic wave propagation effects if the time and space scales are sufficient?

- Such an algorithm if built for an ALE modeling framework and a mimetic based numerical method would required some cross deRham diagram linked algorithmic characteristics.

# Maxwell Equations and Continuum Mechanics

- Kovetz

$$
\begin{aligned}
\nabla \times \mathcal{H} &= \mathcal{J} + \overset{*}{\mathbf{D}}, \\
\nabla \cdot \mathbf{D} &= q, \\
\nabla \times \mathcal{E} &= -\overset{*}{\mathbf{B}}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P}, \\
\mathcal{H} &= \mu_0^{-1} \mathbf{B} - \mathbf{v} \times \epsilon_0 \mathbf{E} - \mathcal{M}
\end{aligned}
$$

- Constitutive theory provides $\mathcal{M}$, $\mathbf{P}$ and $\mathcal{J}$ with $\mathcal{E} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$
- Flux derivatives

$$
\begin{aligned}
\overset{*}{\mathbf{B}} &= \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) + \mathbf{v}(\nabla \cdot \mathbf{B}) = \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) \\
\overset{*}{\mathbf{D}} &= \frac{\partial \mathbf{D}}{\partial t} + \nabla \times (\mathbf{D} \times \mathbf{v}) + \mathbf{v}(\nabla \cdot \mathbf{D}) = \frac{\partial \mathbf{D}}{\partial t} + \nabla \times (\mathbf{D} \times \mathbf{v}) + q\mathbf{v}
\end{aligned}
$$

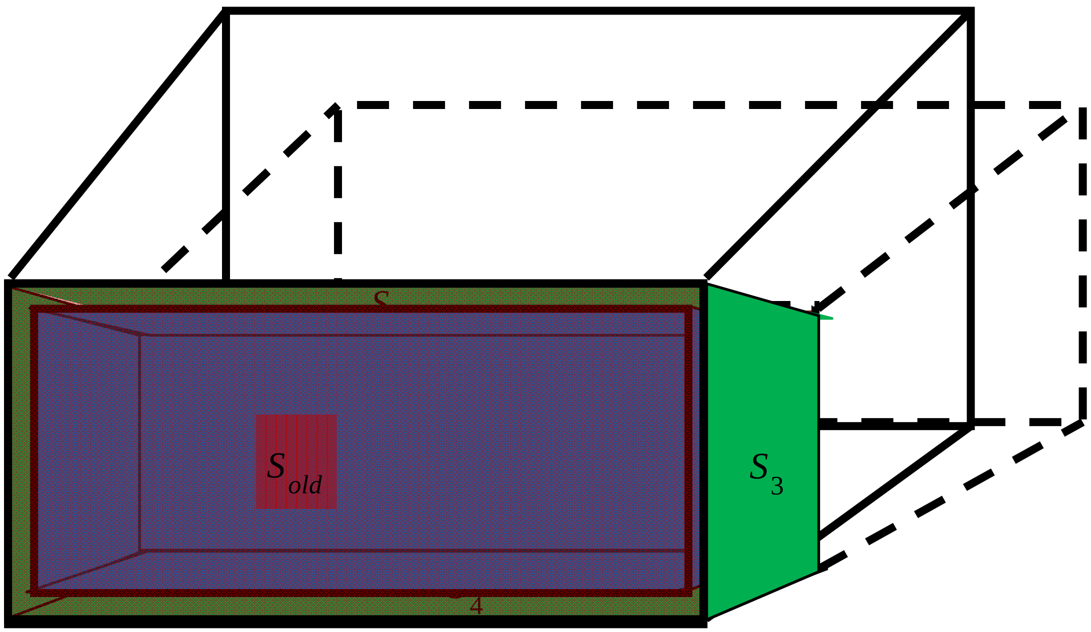
- Fundamental equations still up for discussion, e.g. Weile, Hopkins, Gazonas and Powers, "On the proper formulation of Maxwellian electrodynamics for continuum mechanics," Continuum Mech. Thermo., DOI 10.1007/s00161-013-0308-7.

# Possible Solution

- Take a page from 3D ALE MHD and place D and B as fundamental variables (fluxes) on faces using face elements.
- Operator split the Lagrangian step.
- Mesh motion occurs with constant D and B fluxes. This conserves both the zero magnetic flux divergence property and charge.
- Update the fluxes and electric displacements using a mimetic method.
  - The Bochev and Gunzberger algorithm, "Least-Squares Finite Element Methods," p.225 is a good candidate.
  - Use an L stable time discretization method.
- Remap magnetic flux using standard constrained transport.
- What about remap of electric displacement?

# CT plus a volume term!

$$\frac{\partial \mathbf{D}}{\partial t} + \boxed{\nabla \times (\mathbf{D} \times \mathbf{v})} + \boxed{\mathbf{v}(\nabla \cdot \mathbf{D})}$$




New electric displacement flux is the oriented sum of edge contributions which does not change the charge plus face flux contributions which do.

# ALE Multiphysics and the deRham Complex

- There are many opportunities to use geometrically based methods associated with the deRham complex in ALE multiphysics modeling.
- The three integral transport theorems essential to two-step ALE methods provide fundamental meaning.
- The ideas associated with numerical methods tend to be intuitive and natural.
- Many opportunities are available for additional advances in robustness, computational speed, accuracy, extended modeling and fundamental understanding.