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Physical Foundations of Landauer's Principle

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Introduction

- **Context:** Landauer's Principle connecting information loss with entropy increase is a key motivation for reversible computing, but continues to be frequently misinterpreted/misunderstood.
  - As a result of these misunderstandings, the validity of Landauer's Principle has often been (misguidedly) challenged by engineers & physicists…
    - This has been a substantial barrier to R&D investment in reversible computing
  - We should all be well prepared to answer in-depth questions about the Principle, so we are better equipped to help defend & promote our field.
- There are a number of important subtleties that must be appreciated in order to have an understanding of Landauer's principle that allows one to answer questions about it properly:
  - Transformations of complex states – Focus of a paper at ICRC'16
  - Role of conditional reversibility – Focus of my RC'17 paper
  - **Treatment of stochastic operations** – Addressed in the current paper
    - I also first mentioned this issue many years ago, *e.g.*, at ISMVL'05
  - **Importance of correlations** – Addressed in the current paper
- Landauer's Principle follows as a rigorous theorem of fundamental physics, but *only* given a proper treatment of these issues.

Talk Outline

- Review of historical development of the concept of entropy
  - Shows how information theory emerged from, and is intimately connected with, statistical physics
- Review of some basic information theory concepts:
  - Entropy, known information, conditional entropy, mutual information
- Fundamental connections between computation and physics:
  - Bijective time evolution and the second law of thermodynamics
  - Relationship between computational and physical states
  - Types of computational operations, and thermodynamic implications
    - Many-to-one operations, and implications for entropy ejection
    - Stochastic (one-to-many) operations, and implications for entropy intake
      - Some physical examples
  - Essential role of mutual information in proving Landauer's Principle
- Review of empirical demonstrations of Landauer's Principle
  - Not needed if you know the physics, but helpful in countering skepticism
- Conclusion

A brief history of entropy… (1/5)

- Clausius (1862) identified a quantity $\Delta Q/T$ (with ΔQ = change in heat, T = temperature) that is always ≥ 0 when summed over all of the systems involved in any given thermodynamic process…
  - An early version of the Second Law of Thermodynamics
- In 1865 he proposed to call this quantity *entropy*, from Greek τροπή (tropḗ, transformation),
  - Connoting, that which gives an inherent (en-) direction to a physical transformation (tropḗ)…
  - He also introduced the use of the symbol S for it
    - Possibly to honor Sadi Carnot? (The inventor of the concept of a thermodynamically reversible heat engine)
  - **Note:** Entropy S has *physical* units of *heat/temperature.*
- Nowadays, we actually *define* temperature T in terms of the marginal change in the (maximum) entropy $\hat{S}$ of the system as heat is added to it…

Rudolph Clausius
Discoverer of entropy

$$\frac{1}{T} = \frac{\partial \hat{S}}{\partial Q}$$

T = Thermodynamic temperature
$\hat{S}$ = Maximum entropy (S at equilibrium)
Q = Quantity of heat

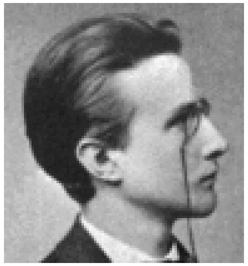
(continued on next slides…)

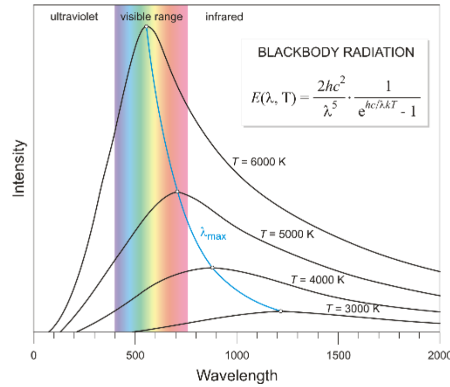
History of Entropy, cont. (2/5)

- Ludwig Boltzmann (1872) proved his *H-theorem* suggesting that entropy might have a *statistical* basis:
  - This paper was essentially an early exploration into what we now call *chaos theory*…
    - Boltzmann showed that statistical uncertainty about the states of common (classical) physical systems (*e.g.* gas molecules) tends to become amplified when those systems interact (*e.g.*, collide)…
  - This paper defined an abstract quantity $H = \int f \log f$ that, in essence, quantified the *degree of certainty* of (or *amount of knowledge* implicit in) any (continuous) probability density function f.
    - **Note:** This paper deserves significant credit as the historical antecedent (70 years ahead!) of Shannon's entropy H, and the entire field of *information theory!*
    - Boltzmann showed, in his theorem, that H tends to decrease over time as gas molecules collide, and suggested that physical entropy S was related (negatively) to H,
      - but he did not yet know how to derive an exact relation between these quantities…

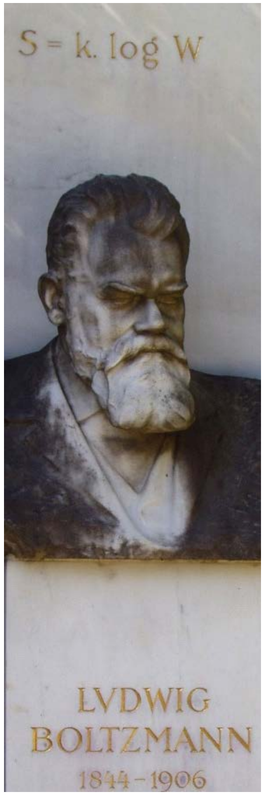
Ludwig Boltzmann
Sire of statistical mechanics, "great-grandfather" of chaos theory & information theory!

History of Entropy, cont. (3/5)





- Max Planck (1901) inferred, via his study of blackbody spectra, that physical states must be *quantized*,
  - As opposed to the *continuous* state variables used in classical mechanical models…
- *I.e.*, you can *count* a (finite) number of distinct states!
  - This let Planck finally give Boltzmann's intuition (that entropy is a statistical quantity) a precise quantitative foundation.
- Planck was first to calculate the magnitude, in physical units, of a natural logarithmic quantum of entropy k:
  - $[\log \mathrm{e}] = k = k_{\mathrm{B}} \cong 1.38 \times 10^{-23}\ \mathrm{J/K}$
    - Called "Boltzmann's constant" to honor Boltzmann's earlier role
- Defining k physically lets us write $\hat{S} = \log W = k \ln W$.
  - Rendered as $S = k \log W$ on Boltzmann's tombstone… →
    - The separate quantity H was, at this point, no longer needed…



History of entropy, cont. (4/5)

- John von Neumann developed not only the ENIAC's "von Neumann" architecture, but also (among many other things) the mathematical formulation of quantum mechanics…
  - Unifying the Schrödinger and Heisenberg formalisms
- In a 1927 paper, he showed how to translate the Boltzmann-Planck entropy concept (for the general case of nonuniform probabilities) to the language of quantum states: $S = -k\,\mathrm{Tr}\,(\rho \ln \rho)$.
  - ρ = density matrix; Tr = matrix trace operator
- This is (exactly) just $S = -k \sum p \ln p$, where the p are the probabilities of the pure quantum states that the ρ matrix represents a statistical mix of.
  - If you already know about Shannon entropy, this formula should be starting to look awfully familiar…

John von Neumann
Formulated the modern concept of entropy

History of entropy, cont. (5/5)

- Shannon (1948) is usually credited with "inventing the information-theoretic concept of entropy,"

$$H = -\sum p \log_2 p \ \text{ bits},$$

- However, in that formula, Shannon was really just *reformulating and reapplying* already-existing concepts that were already very well-established in physics:
  - **Note:** Shannon explicitly *cites* Boltzmann's contribution!
    - The symbol H, and the use of the expected log-probability quantity come straight out of Boltzmann's 1872 H-theorem!
      - The sign of H here is just changed to match that of S
    - Also, the transition from Boltzmann's continuous $\int$ to the discrete Σ case had <u>already been completed</u> over the period 1901-1927 (21 years prior!) by Planck and von Neumann.
  - Further, the change of the entropy unit from k to bit only reflects a shift in the conventional choice of the logarithmic unit from log e to log 2, nothing more.
- So really, the only true innovation in Shannon's entropy concept was:
  - The states that Shannon was explicitly concerned with in his work were not microscopic *physical* states, but macroscopic *digital* or symbolic states.
    - Yet, Shannon's entropy connects fundamentally to physical entropy, as we'll see…

Claude Shannon
Applied entropy in communication theory

Entropy in a Nutshell

Basic review + coining some useful terminology

- Define the "surprisingness" or *surprise* $s(x)$ of any event x that has a 1 in m chance of occurring as $s = s(x) = s(m) = \log m$.
  - Call the $m \geq 1$ "improbability;" it can be a non-integer.
    - s is logarithmic b/c the improbabilities of independent surprises multiply.
  - *Indefinite* logarithm; dimensioned in *arbitrary* logarithmic units.
    - Some example units: $\log 2 = 1$ bit; $\log \mathrm{e} = 1$ nat $= k_{\mathrm{B}}$; $\log 10 = 1$ bel.
- In terms of event's *probability* $p = p(x) = p(m) = 1/m$,

$$s(p) = \log\frac{1}{p} = -\log p\,.$$

- Define event's psychological "*heaviness*" $h = h(x) = h(p)$ as its surprise, weighted by its probability:

$$h(p) = s/m = p \cdot s = p\log m = -p\log p.$$

- Then for any probability distribution $p(x)$ over any mutually exclusive and exhaustive set of events $\mathbf{X} = \{x_1, \ldots, x_n\}$, we have that the ***expected surprise*** $S(X) = \mathrm{E}_p[s(x)]$ and the ***total heaviness*** $H(X) = \sum_{x\in X} h(x)$ associated with that particular set of possible events are the same, and are given by:

$$S(X) = \sum_{x\in X} p(x)\cdot s(x) = H(X) = -\sum_{x\in X} p(x)\cdot \log p(x)\,.$$

- We call this quantity $H = S$ the *entropy* of the given epistemic situation.
  - By convention, we'll prefer H for "computational" entropy, S for "physical" entropy.

Improbability:

$m = 6\times 6 = 36$

Surprise:

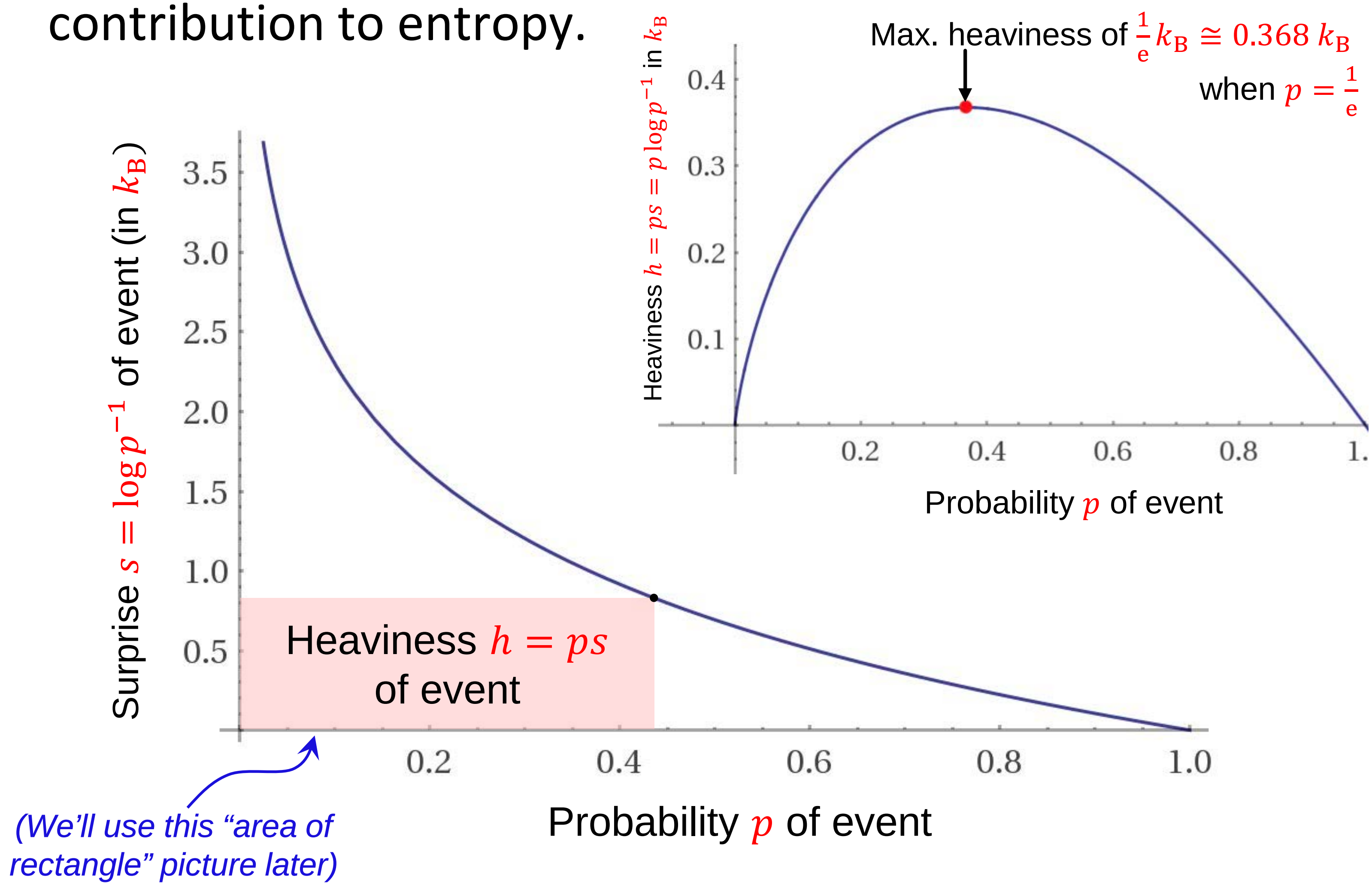
$s = 2(\log 6)$

Heaviness:

$h = \frac{s}{m} = \frac{2}{36}\log 6$

Surprise and Heaviness Functions

- For an individual state's contribution to entropy.



Known Information & Information Capacity

- Given a discrete variable V with a state set $\mathbf{V}$, and a probability distribution $P(V)$,
  - The *amount of known information* $K(V)$ about V is given by the maximum entropy $\widehat{H}$ minus the entropy H:

$$\begin{aligned} K(V) &= \widehat{H}(V) - H(V) \\ &= \log|\mathbf{V}| - H(V) \end{aligned}$$

  - The *information capacity* $\mathbf{I}(V)$ of V is the maximum known information or entropy, and is also the sum of known information and entropy.

$$\begin{aligned} \mathbf{I}(V) &= \widehat{K}(V) = \widehat{H}(V) = \log|\mathbf{V}| \\ &= K(V) + H(V) \end{aligned}$$

Conditional Entropy & Mutual Information

- Given a discrete variable V with a state set $\mathbf{V}$ expressible as a cartesian product $\mathbf{X} \times \mathbf{Y}$ of two state sets $\mathbf{X}, \mathbf{Y}$ for variables X, Y, and a joint probability distribution $P(X,Y)$,
  - The *conditional entropy* $H(X \mid Y) = H(X,Y) - H(Y)$.
    - Expected value of $H(X)$ that would result from learning the value of Y.
  - The *mutual information* is a symmetric function given by:

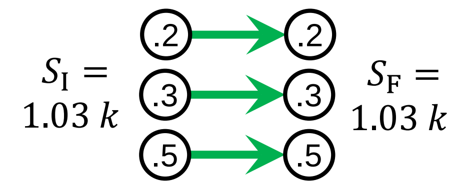
$$\begin{aligned} I(X;Y) = I(Y;X) &= K(X,Y) - K(X) - K(Y) \\ &= H(X) + H(Y) - H(X,Y) \\ &= H(X) - H(X \mid Y) \\ &= H(Y) - H(Y \mid X). \end{aligned}$$

    - The amount of shared/redundant information between X and Y.
      - The degree of information-theoretic *correlation* between the variables.

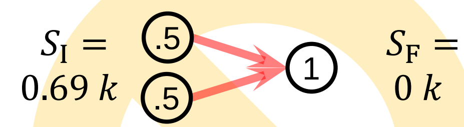
Entropy and Time Evolution

- Microscopic dynamics is one-to-one (injective).
  - A consequence of unitary quantum time evolution.
  - If we could track fully-detailed physical time evolution perfectly, we would see no entropy increase!
    - Probability distribution unchanged, just on new states
- In fact, this *reversibility* of microphysics <u>underlies</u> the Second Law of Thermodynamics.
  - If physics was not injective, entropy could decrease!
- But, entropy can be seen to *increase*, from our subjective perspective as modelers if we have any uncertainty about the microscopic dynamics, or cannot keep track of it in detail…
  - Thus, entropy increase only exists as a subjective epistemological phenomenon…
    - It is always fundamentally just a reflection of our degree of *ignorance & incompetence* in modeling the world…

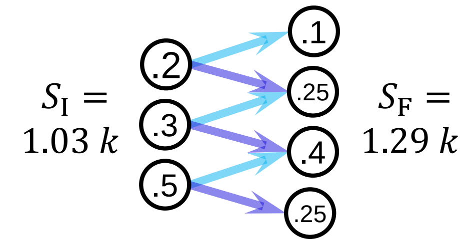
$$S[p] = \mathrm{E}_p[\log p^{-1}]$$



Bijective microphysics → No “true” entropy change



Irreversible microphysics → Entropy would decrease (Second Law of Thermo. would be violated)



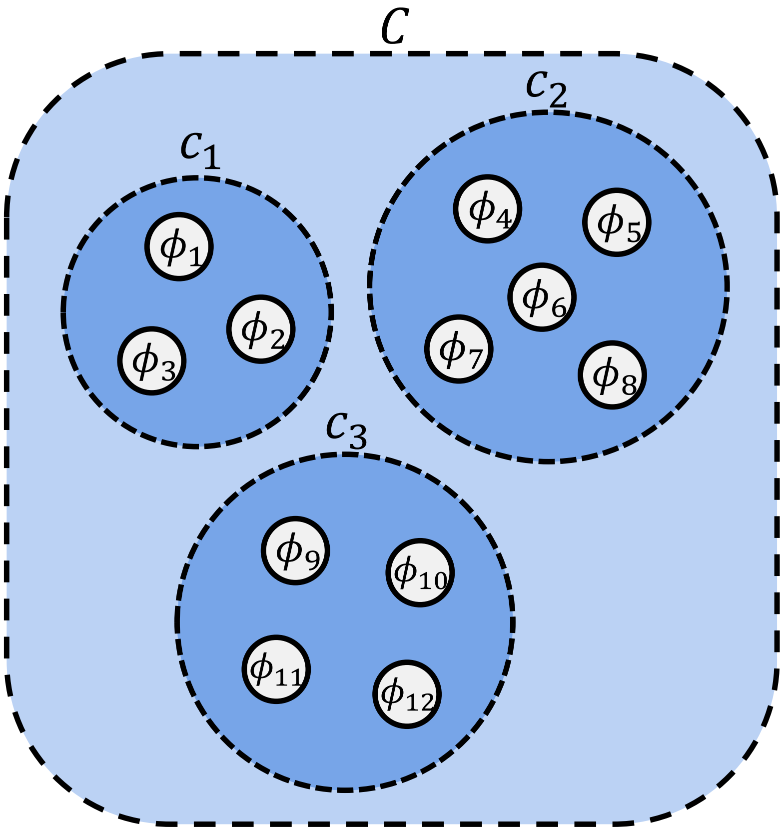
True dynamics uncertain (or not tracked in detail) → Entropy increases

From Physics to Computation

- Thermodynamics and quantum mechanics show that any bounded physical system admits only a finite set $\Phi = \{\phi_1, \ldots, \phi_n\}$ of measurably distinguishable detailed physical states (*microstates*).
  - *E.g.*, Φ could be any orthogonal set of basis vectors for the system's Hilbert space.
- We can *group* these microstates, that is, partition them into subsets c_j of micro-states that we consider as *equivalent* to each other for some designated purpose…
  - *e.g.*, for purposes of representing some specific *computational* information
- Any probability distribution $p(\phi_i)$ over the physical state space Φ induces a probability distribution P over the computational state space (subsystem) $C = \{c_j\}$ as well…

$$P(c_j) = \sum_{\phi_i \in c_j} p(\phi_i).$$

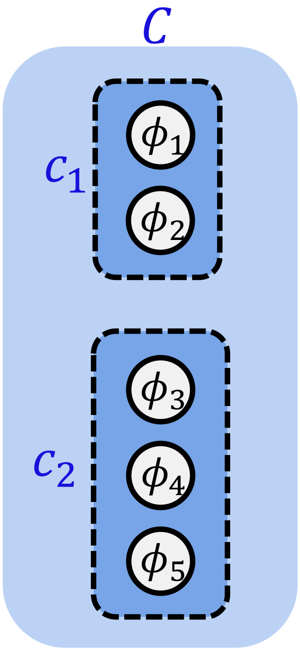
- This implies a *computational entropy* $H(C)$.

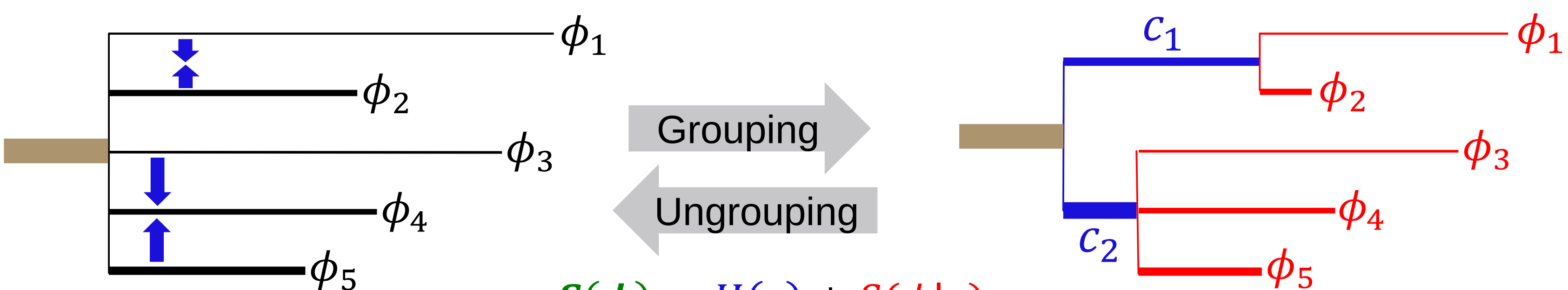


Example of a computational state space C consisting of 3 distinct computational states c_1, c_2, c_3, each defined as a set of equivalent physical states.

Visualizing Entropy of Grouped States

- Can represent a hierarchy of events in a tree structure…
  - Branch ***thickness*** = event probability p.
  - Branch ***length*** = *incremental surprise* Δs associated w. event,
    - relative to whatever base event it's branching off from.
  - Branch ***area*** = event's *incremental heaviness* $\Delta h = p\Delta s$, *i.e.*,
    - its contribution to total entropy, in addition to its base event's.
- *Grouping* events into larger events has these effects:
  - Thicknesses (probs.) of branches combine in parent branch
  - A corresponding part of the total length (surprise) of each branch is reassociated to parent (stem) branch.
  - Note: The total heaviness $\boldsymbol{H}$ of all branches and stems (total entropy ***S***) is not changed at all by any grouping/ungrouping!!

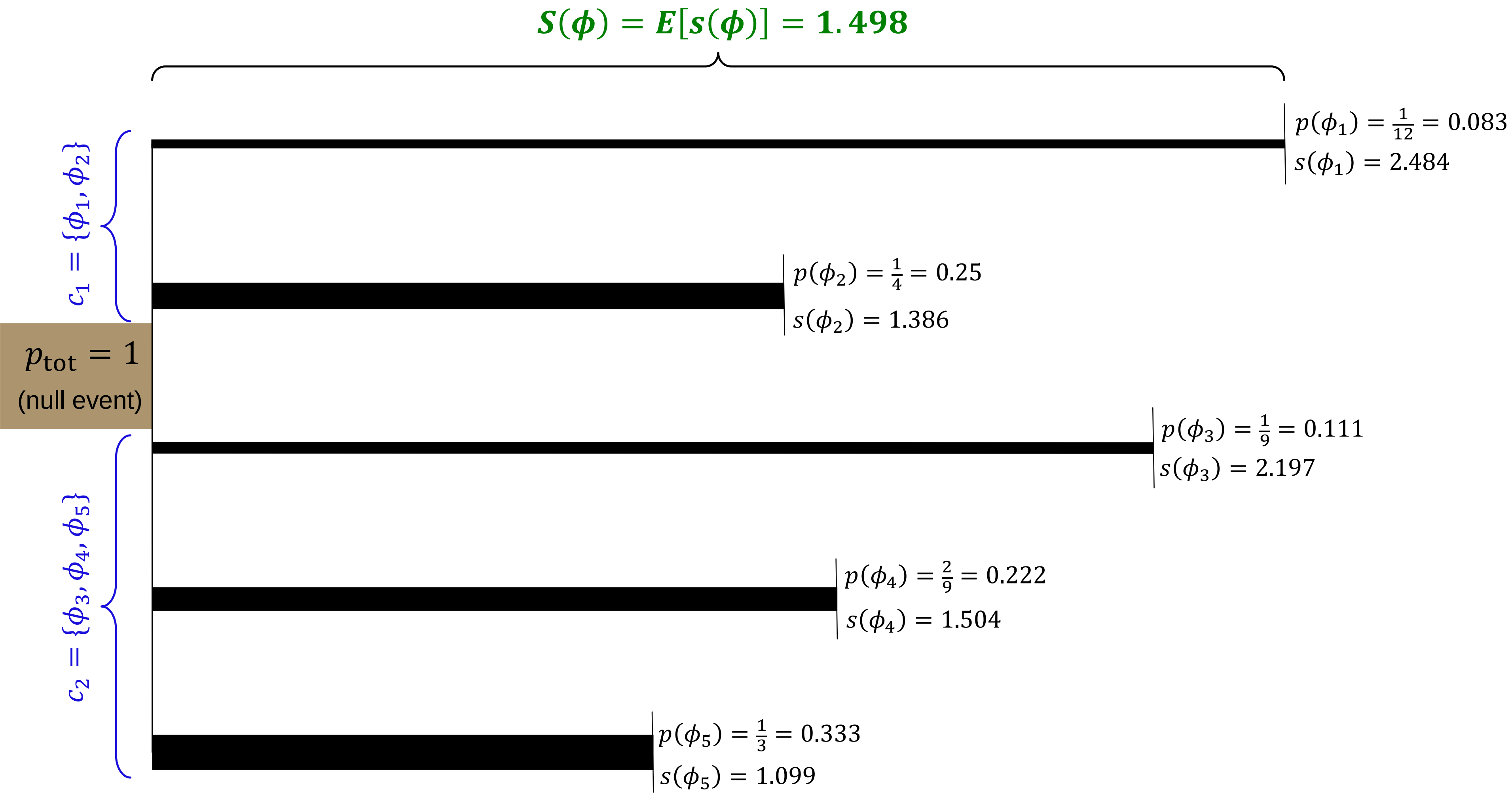




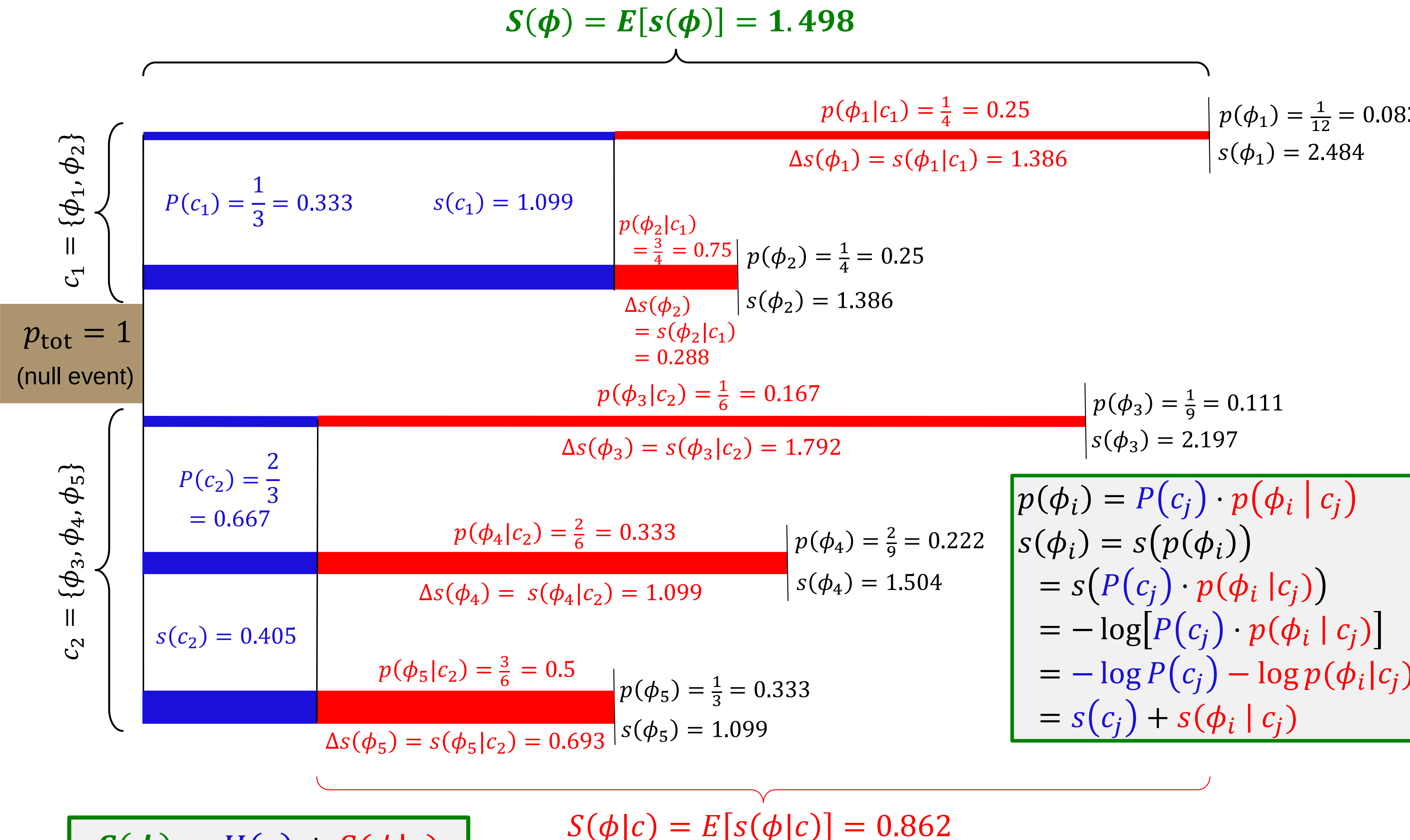
$$\boldsymbol{S}(\boldsymbol{\phi}) = H(c) + S(\phi|c)$$

Total system entropy = computational entropy + non-computational entropy

Grouping of States (slide 1 of 3)



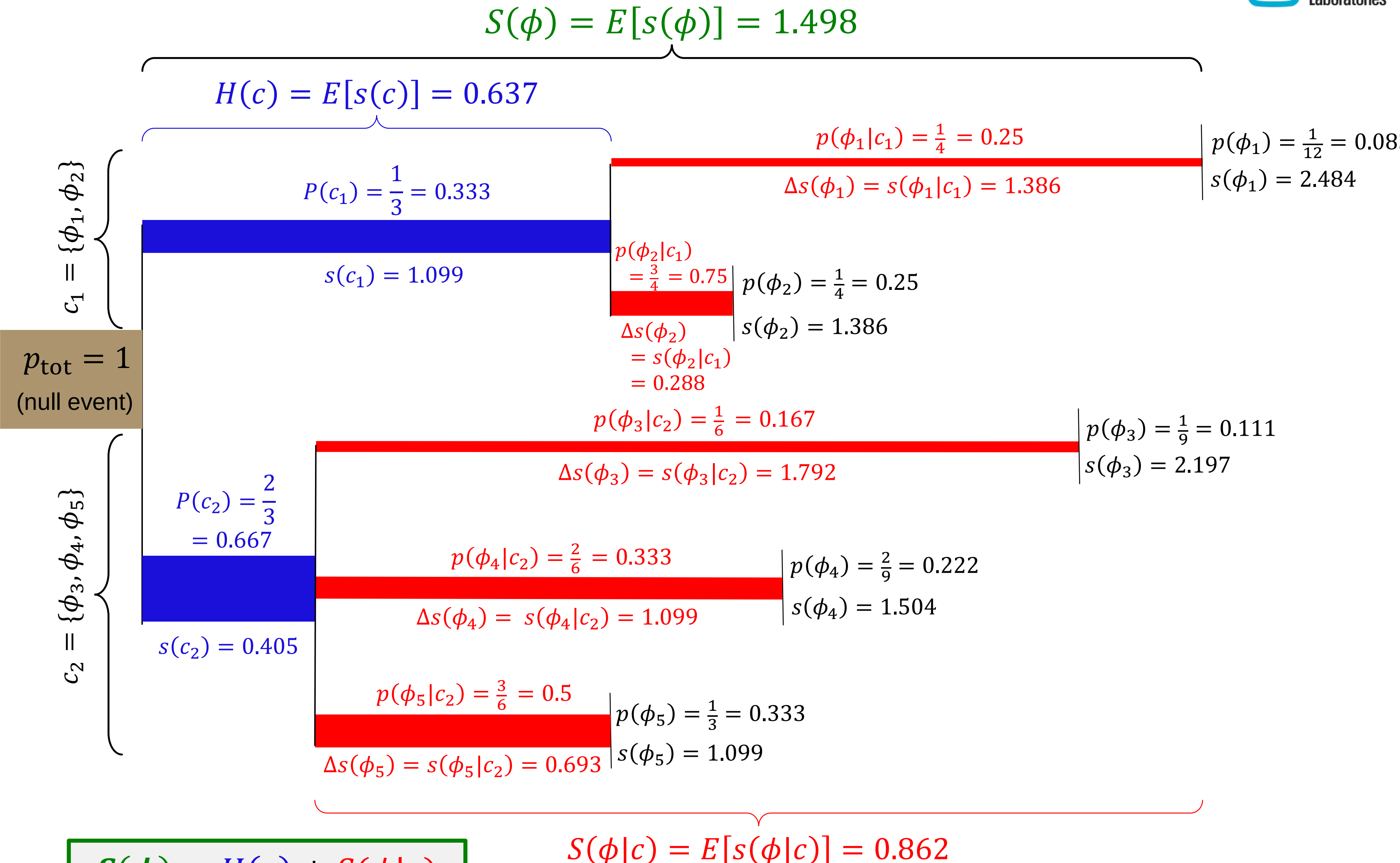
Grouping of States (slide 2 of 3)



$$
\begin{aligned}
p(\phi_i) &= P(c_j) \cdot p(\phi_i \mid c_j) \\
s(\phi_i) &= s(p(\phi_i)) \\
&= s\big(P(c_j) \cdot p(\phi_i \mid c_j)\big) \\
&= -\log\big[P(c_j) \cdot p(\phi_i \mid c_j)\big] \\
&= -\log P(c_j) - \log p(\phi_i|c_j) \\
&= s(c_j) + s(\phi_i \mid c_j)
\end{aligned}
$$

$$\boxed{S(\phi) = H(c) + S(\phi|c)}$$

Grouping of States (slide 3 of 3)



$$\boldsymbol{S(\phi) = H(c) + S(\phi|c)}$$

Total system entropy = computational entropy + non-computational entropy

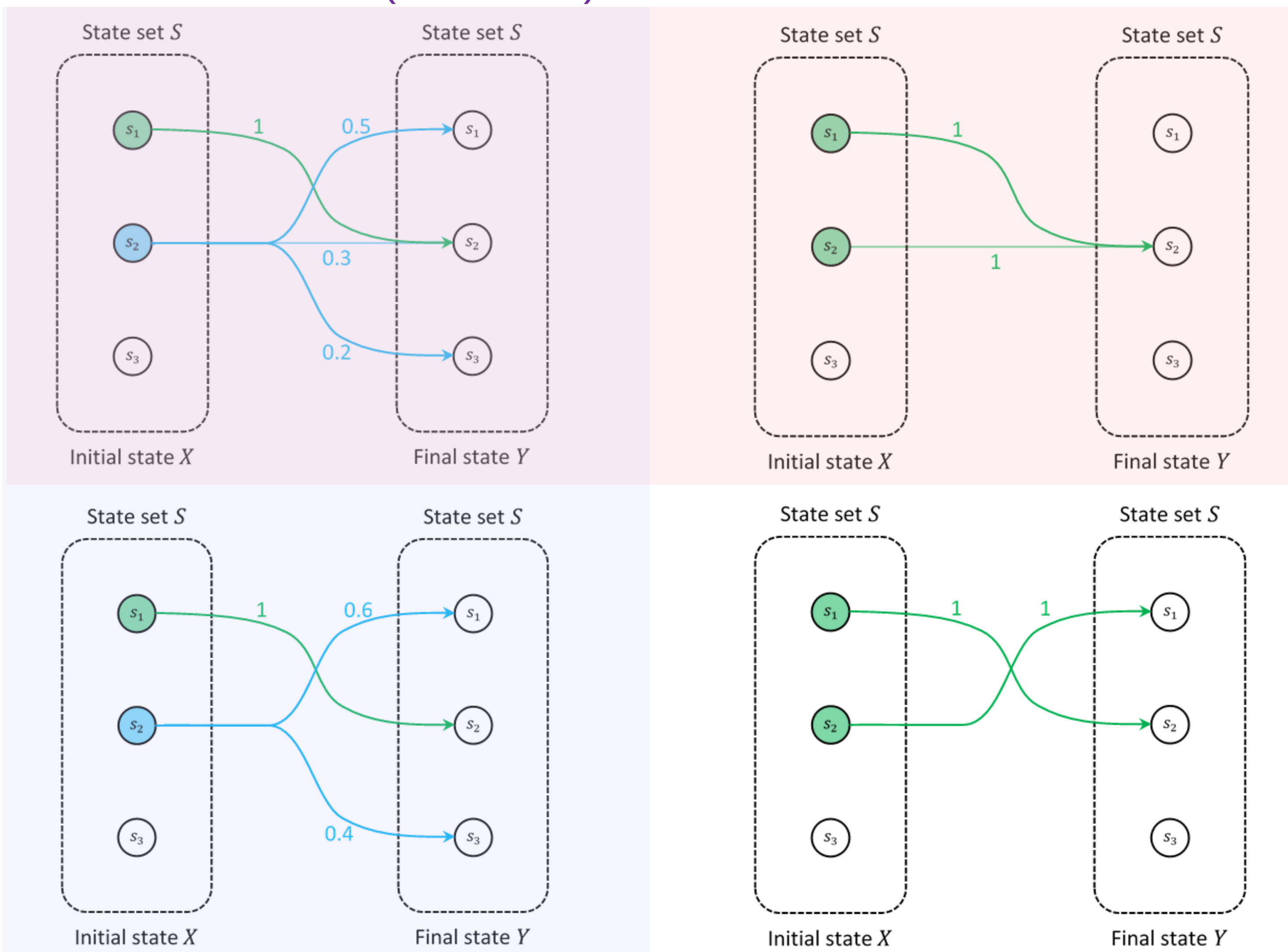
Types of Computational Operations

Define operations as (possibly partial) probabilistic transition relations

Nondeterministic (stochastic) | Deterministic

Irreversible (many-to-one)

(Unconditionally) Reversible (injective)

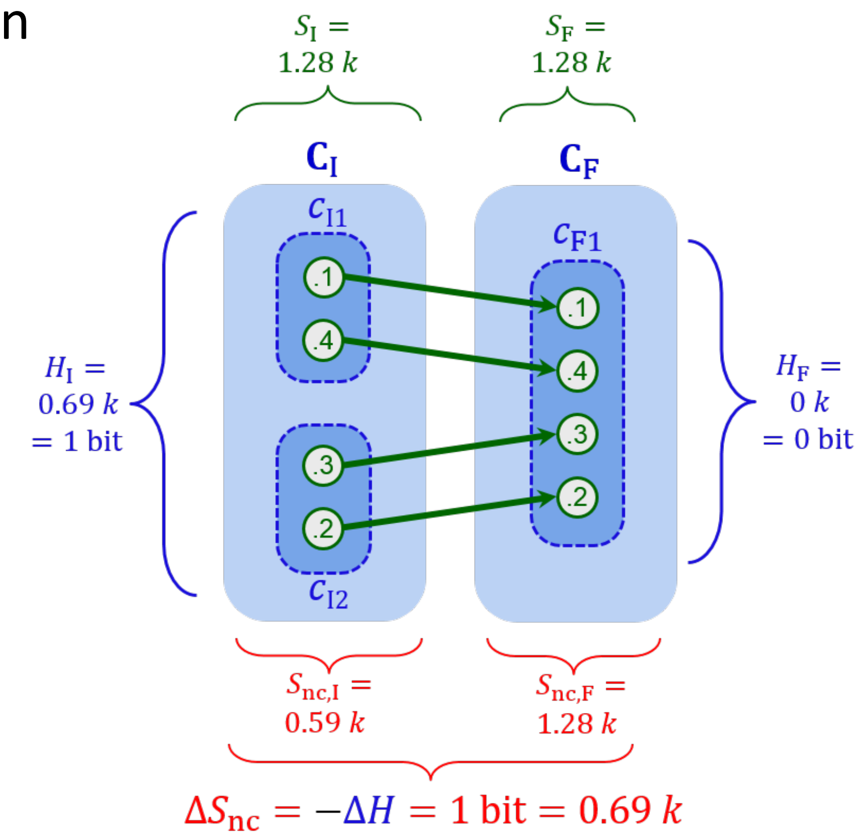


Entropy Ejection in Many-to-One Operations

- Again, a *computational state* c_j is just an equivalence class of physical states ϕ_i
  - On the left we see two computational states c_0, c_1, each with probability 0.5
- The computational subsystem has an induced information entropy $H(c)$.
  - Here, it is $H(c) = \log 2 = 1\text{ bit} = k \ln 2$.
- The *non*-computational subsystem (everything else) has expected entropy

$$S_{\mathrm{nc}} = S(\phi|c) = S(\phi) - H(c) = S - H$$

  - The conditional entropy of the physical state ϕ, given the computational state c.



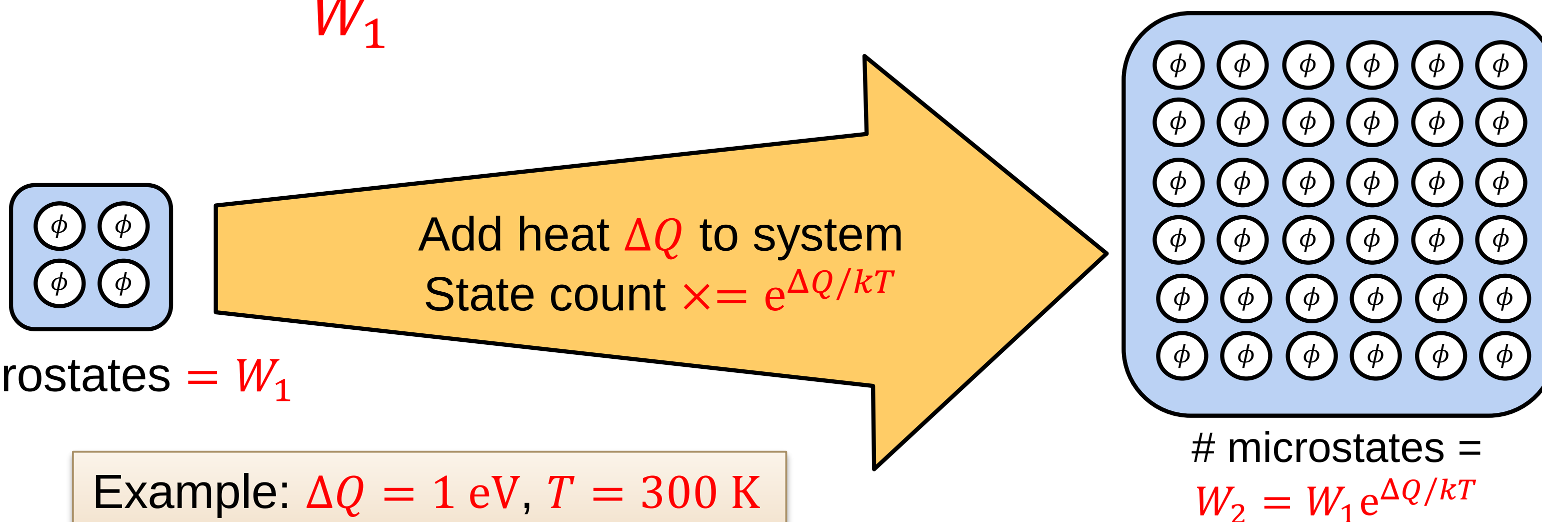
- Thus, if the computational entropy <u>*decreases*</u> (note here $\Delta H = -1\text{ bit}$),
  - The *non*-computational entropy *must <u>increase</u>* by $\Delta S_{\mathrm{nc}} = -\Delta H$ (here, $k \ln 2$).
- Thus, ejecting computational entropy $H = 1\text{ bit}$ implies we must add heat $\Delta Q = kT \ln 2$ to an environment at some temperature T.

Heat Increase inflates State Count

- The Boltzmann relation $\hat{S} = k \ln W$, together with the definition of temperature $1/T = \partial\hat{S}/\partial Q$, immediately implies that whenever a quantity of heat ΔQ gets added to a thermal system, its total number of accessible microstates gets multiplied by a factor $\blacksquare W$ that is given by $\mathrm{e}^{\Delta Q/kT}$:

$$\hat{S} = k \ln W \quad \Rightarrow \quad W = \mathrm{e}^{\hat{S}/k}$$

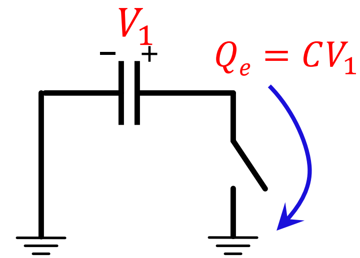
$$\blacksquare W = \frac{W_2}{W_1} = \mathrm{e}^{(\hat{S}_2 - \hat{S}_1)/k} = \mathrm{e}^{\Delta\hat{S}/k} = \mathrm{e}^{\Delta Q/kT}$$



Example: $\Delta Q = 1\ \mathrm{eV}$, $T = 300\ \mathrm{K}$ implies $\blacksquare W \cong 6.3 \times 10^{16}$

Energy Dissipation

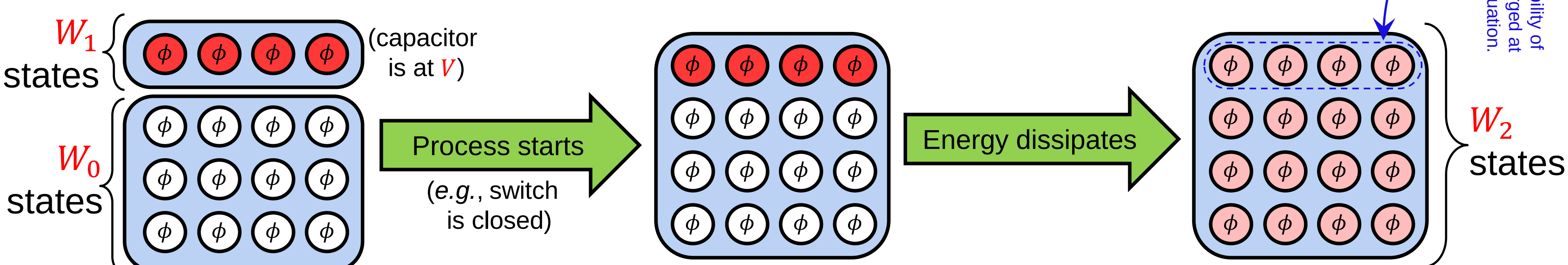
- For *any* means of dissipating some energy E_1 (*e.g.*, by discharging a capacitor), the dissipated energy $E_{\text{diss}} = E_1$ ends up as increased heat $\Delta Q = E_{\text{diss}}$ in a thermal environment at some temperature T…
  - Thus, any means of dissipating energy E_1 results in the same state count multiplier $\blacksquare W = \mathrm{e}^{E_{\text{diss}}/kT} = \mathrm{e}^{E_1/kT}$…



$$E_{\text{diss}} = E_1 = \frac{1}{2}CV_1^2$$

- Thus, we can represent any energy dissipation process as a merging of computational state sets, as follows:
  - Here, red shading indicates concentration of probability mass…

Note there is still a nonzero probability of still finding the capacitor to be charged at V, corresponding to a thermal fluctuation.



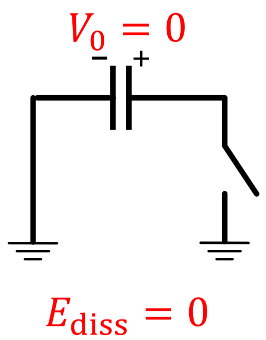
$W_0 = W_2 - W_1$
$= W_1(\mathrm{e}^{E_1/kT} - 1)$

(Before system thermalizes)

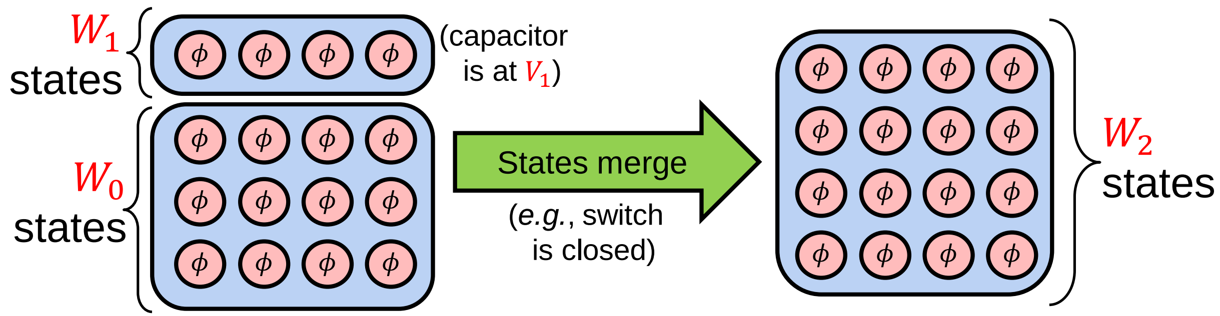
$W_2 = W_1 \cdot \blacksquare W$
$= W_1 \cdot \mathrm{e}^{E_1/kT}$

Entropy increase of $\Delta S = E_1/T$

Asymptotically conditionally reversible many-to-one operations



- In my RC'17 paper, I discussed how entropy ejected to the environment approaches 0 if the probability of <u>all but one</u> of a set of computational states being merged together approaches 0.
  - Corresponds to the case in this circuit example where the capacitor was initially at logic zero (subject to uncertainty due to thermal fluctuations).



$$W_0 = W_2 - W_1 = W_1(\mathrm{e}^{E_1/kT} - 1)$$

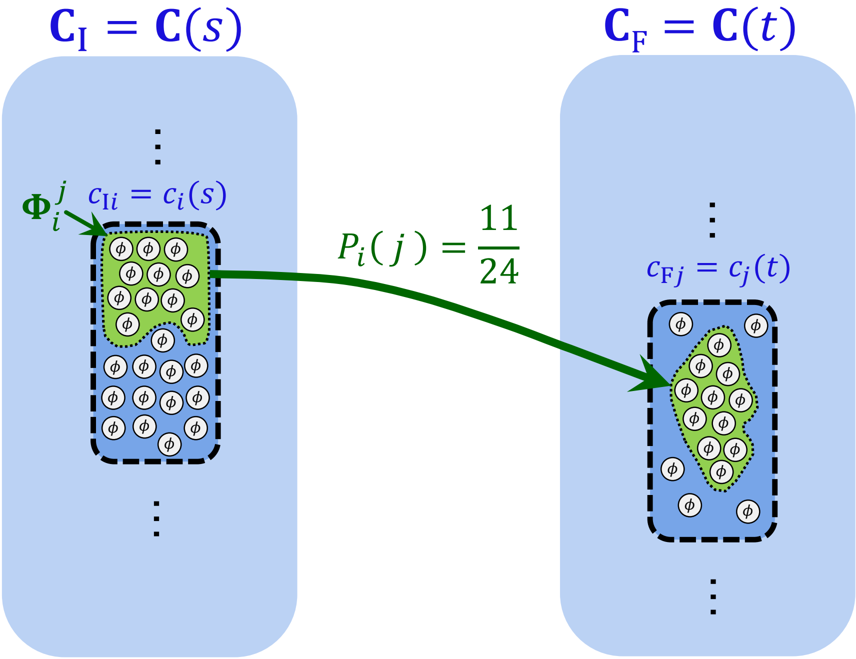
$$W_2 = W_1 \cdot \blacksquare W = W_1 \cdot \mathrm{e}^{E_1/kT}$$

- Here, the initial *computational* entropy is extremely tiny…
  - Prob. of being in logic 1 state initially is only $\mathrm{e}^{-E_1/kT}$, *e.g.*, 3.3×10^{-11} if $E_1 = 0.1$ aJ).
  - Entropy of initial computational state comes out to only $8.2 \times 10^{-10}k$.
    - Thus, only that tiny amount of entropy gets ejected to non-computational form in this case.

Note: Ejection/intake of entropy does not imply net entropy increase! (more on this later)

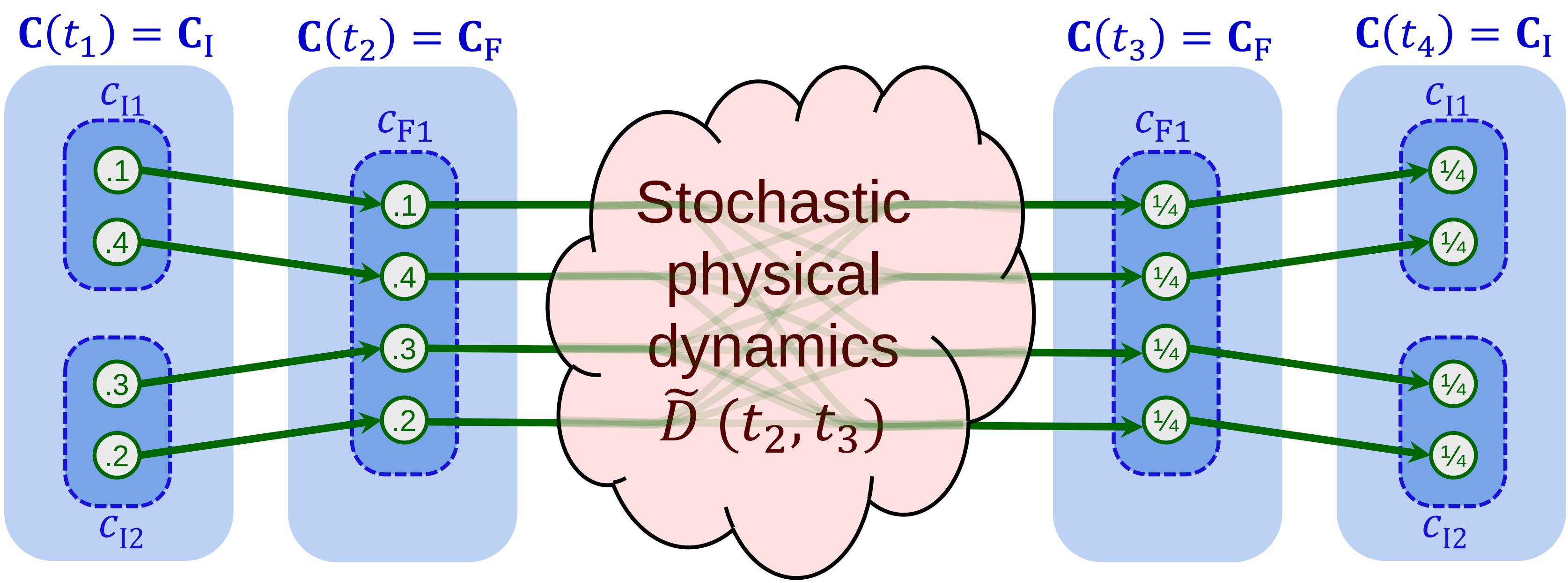
Stochastic computational operations

- These can occur in bijective dynamics if different subsets of the set of physical microstates making up the initial computational state transition to different new computational states…
  - In this example, initial computational state $c_{\mathrm{I}i}$ at time s has probability $P_i(j) = 11/24$ of transitioning to final computational state $c_{\mathrm{F}j}$
    - because 11 out of the 24 (equally-likely) microstates transition into there



Reversing Entropy Ejection (1/2)

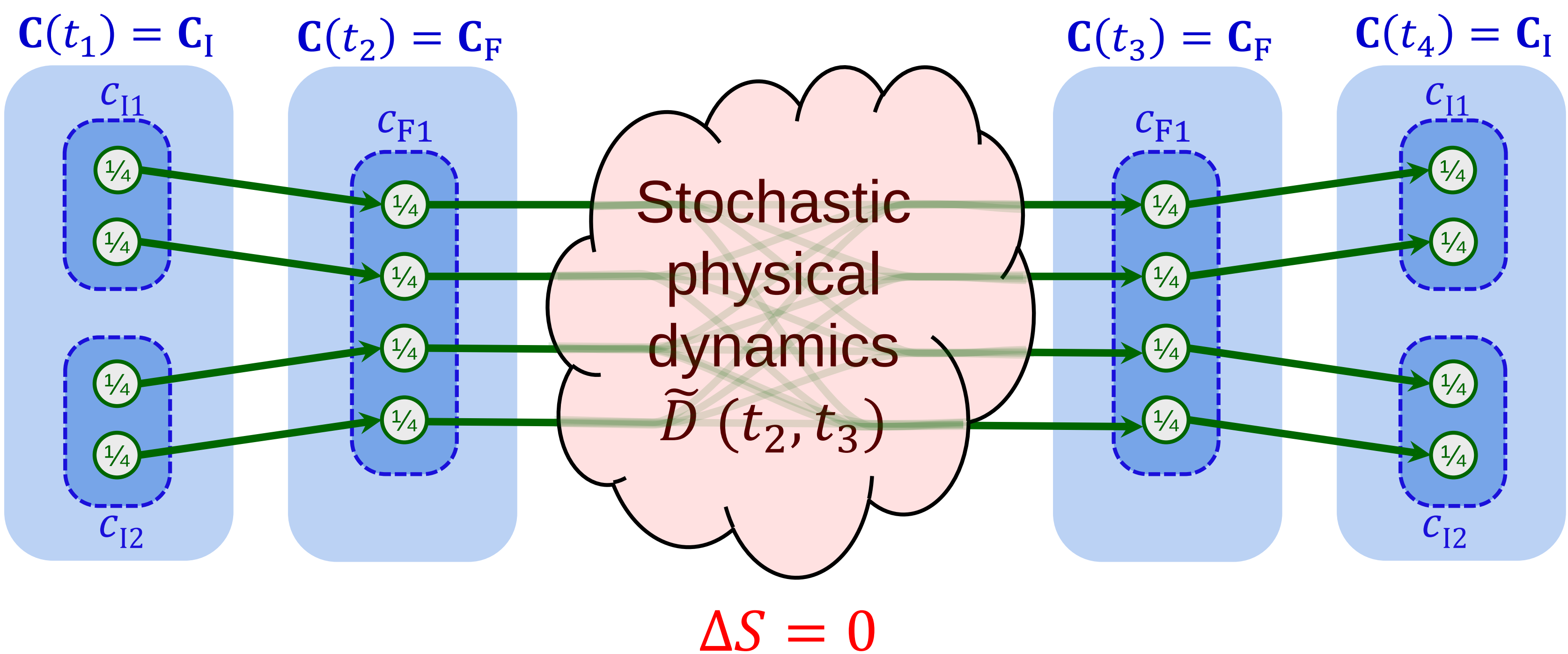
- The ejection of entropy can be reversible!
  - Here, we do a many-to-one operation followed by a stochastic operation to restore the computational entropy back to 1 bit
    - However, in this example, non-computational entropy increases by ~0.15 bits (some information about the initial physical state has been lost)



$$\Delta S = 0.106\ k = 0.153\ \text{bit}$$

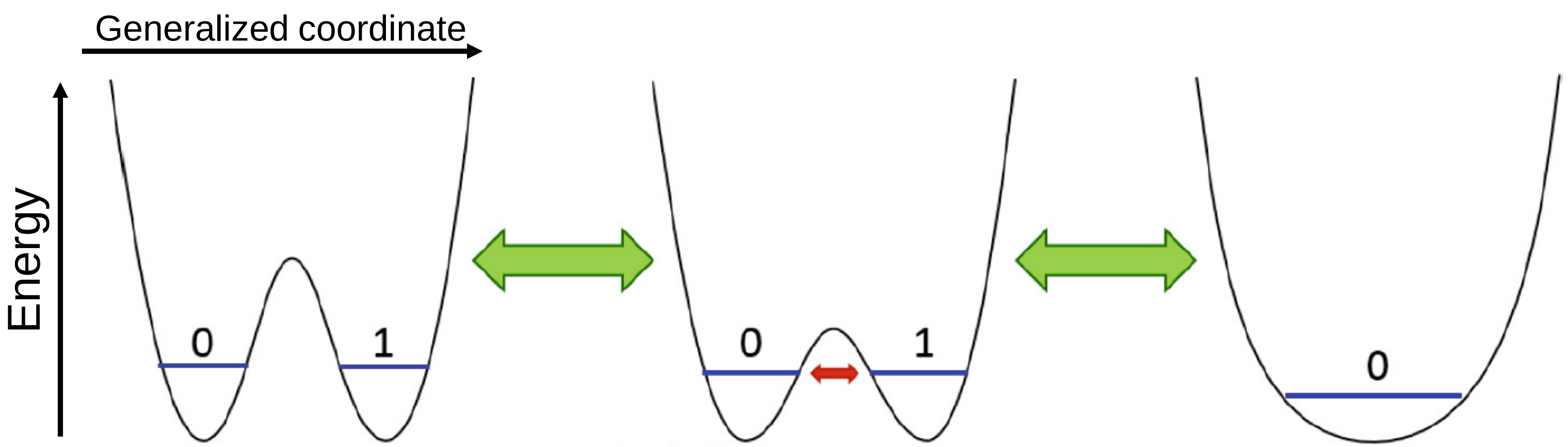
Reversing Entropy Ejection (2/2)

- The ejection of entropy can be reversible!
  - Here, we do a many-to-one operation followed by a stochastic operation to return the computational entropy to 1 bit
    - In ***<u>this</u>*** example, the initial non-computational entropy was *<u>already</u>* maximal, so there is no entropy increase!



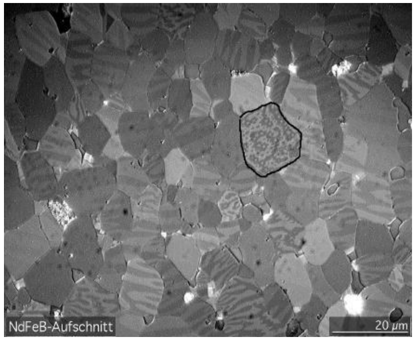
Bistable potential well implementation

- A simple class of physical implementations of the entropy ejection and intake process on the previous slide
  - Use two degenerate states separated by a potential barrier
    - *E.g.*, in quantum dots, superconducting circuits, many other systems
  - Lowering the barrier partially will allow states to equilibrate
  - Lowering the barrier fully will completely merge the states
  - Going left and then right re-randomizes the digital state
    - If done adiabatically, there is no increase in entropy in this process

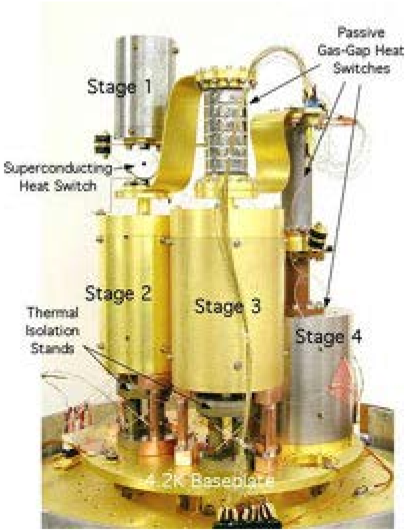


$$\mathbf{C}(s) = \mathbf{C}_{\mathrm{I}} = \{c_{\mathrm{I}0}, c_{\mathrm{I}1}\} \qquad \mathbf{C}\left(\frac{s+t}{2}\right) = \left\{c_0\left(\tfrac{s+t}{2}\right), c_1\left(\tfrac{s+t}{2}\right)\right\} \qquad \mathbf{C}(t) = \mathbf{C}_F = \{c_{\mathrm{F}0}\}$$

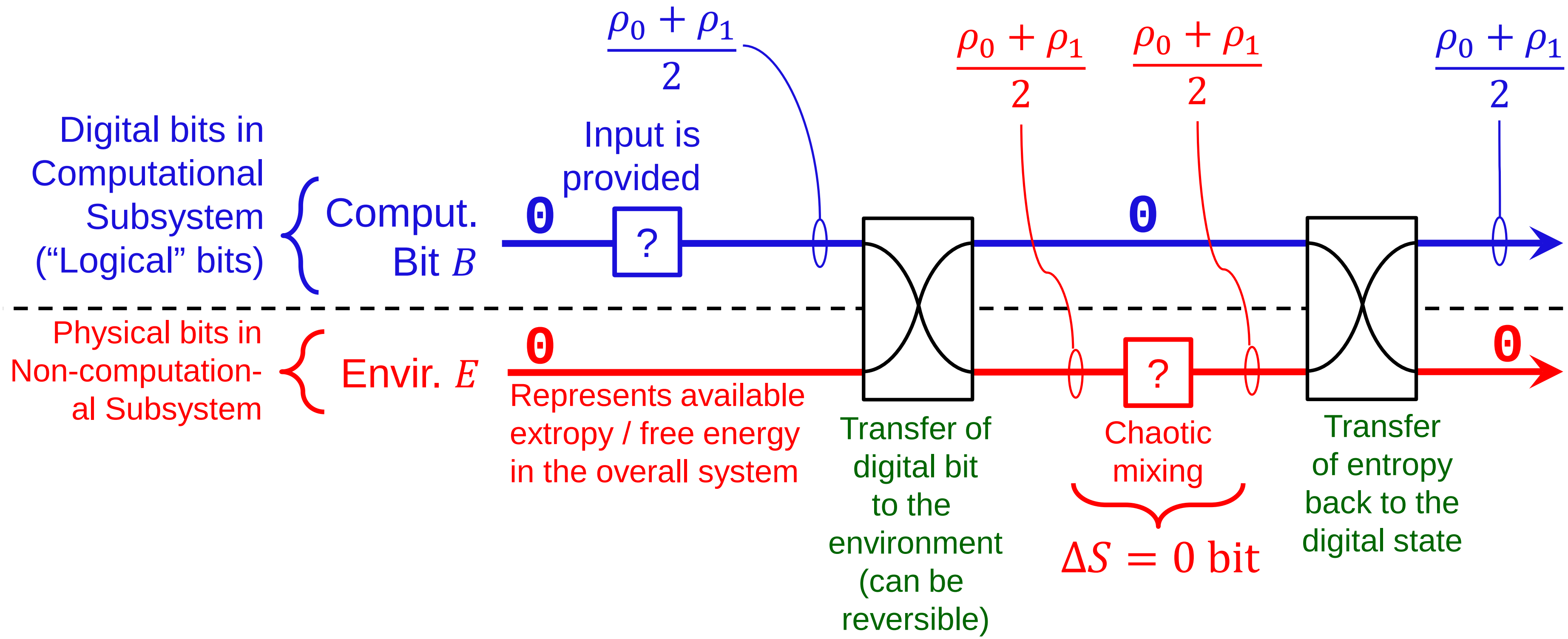
Adiabatic demagnetization / paramagnetic refrigeration

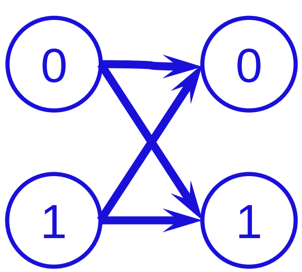


- A physical phenomenon that has been well studied for a very long time, and that nicely illustrates the reversibility of entropy ejection
  - Utilized in practice in cryogenic applications!
- The randomly-oriented magnetic domains in a sample of paramagnetic material can be considered to contain entropy in a "frozen," "digital," "computational" form…
  - When you apply a magnetic field and align the domains, this ejects the entropy from the domains and heats their surrounding environment…
  - But if you *remove* the magnetic field adiabatically, and allow the domains to re-randomize themselves, this *takes in* entropy from the thermal environment, locking it into this "digital" form, and *cools* the thermal environment!



Thermodynamically Reversible Erasure of an Uncorrelated Bit

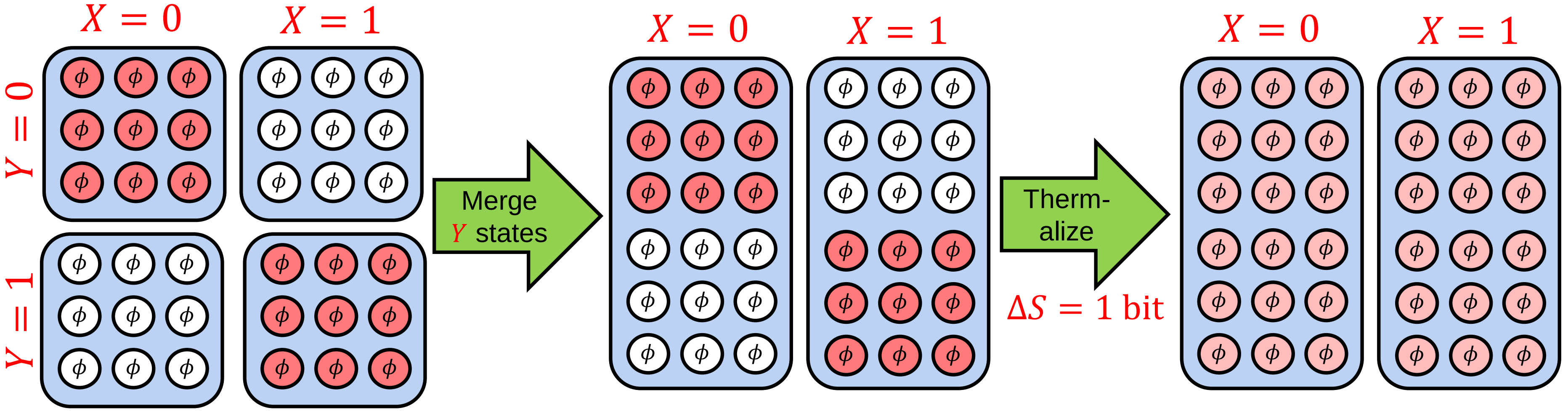




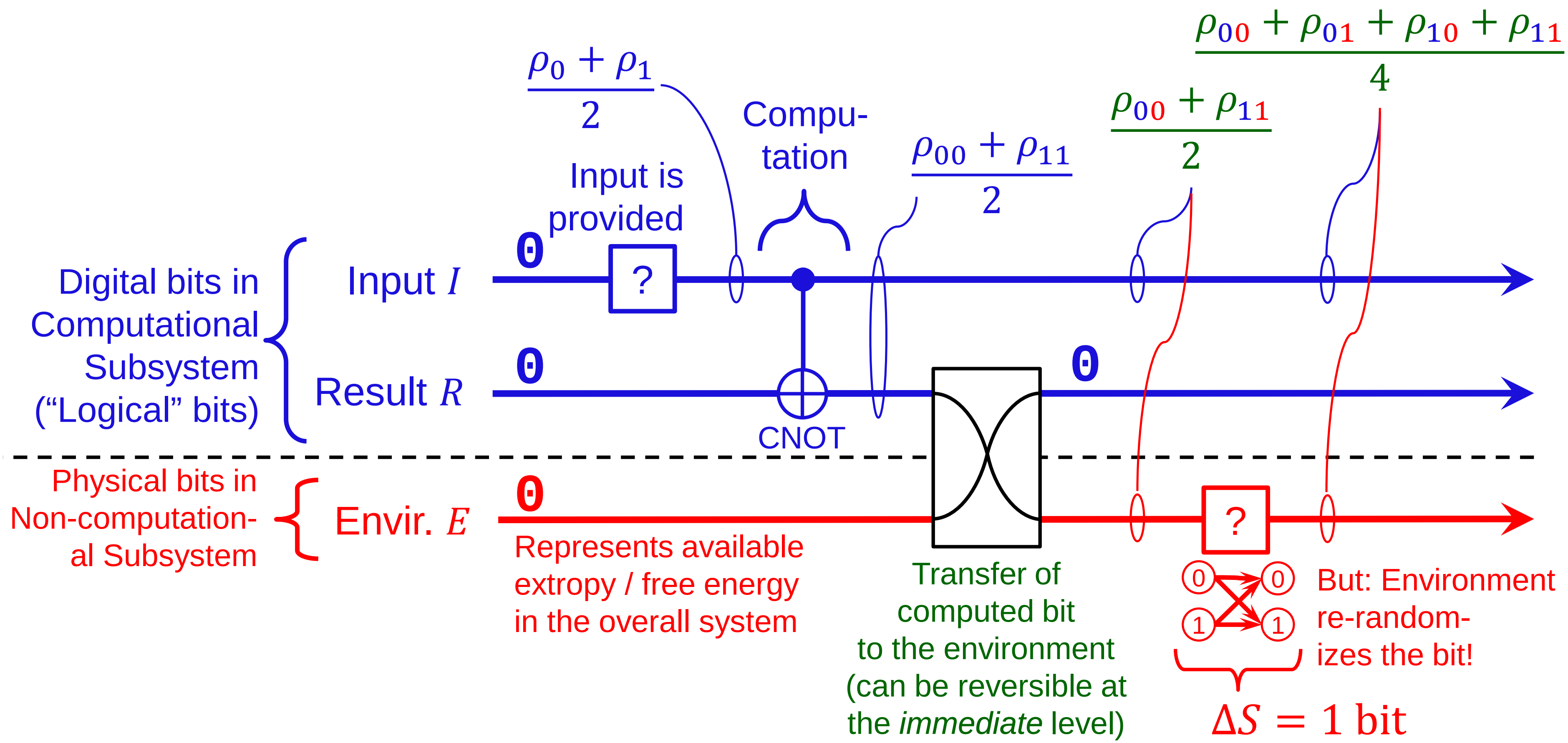
- Overall map including mixing is non-injective
- No autocorrelation between initial & final state
- Not “logically reversible” in traditional sense

Role of Correlations

- In light of the foregoing points, this is essential for understanding the true reason for the entropy increase in Landauer's principle!
  - Suppose we have two one-bit computational state variables, X and Y, and suppose that initially X is *random*, but we know that $Y = X$.
    - *E.g.*, this would be the case if Y was computed earlier using $Y := X$.
  - Thus, the joint system XY contains 1 bit of entropy, but also 1 bit of *known information* that is shared between X and Y (*i.e.*, mutual information).
  - If Y is then erased, this *known* computational information is ejected to become (briefly) known *non-computational* information, which then rapidly becomes thermalized, (permanently) increasing entropy by 1 bit's worth ($k \ln 2$).
    - This is why erasing *computed* bits (in particular) is *not* thermodynamically reversible!

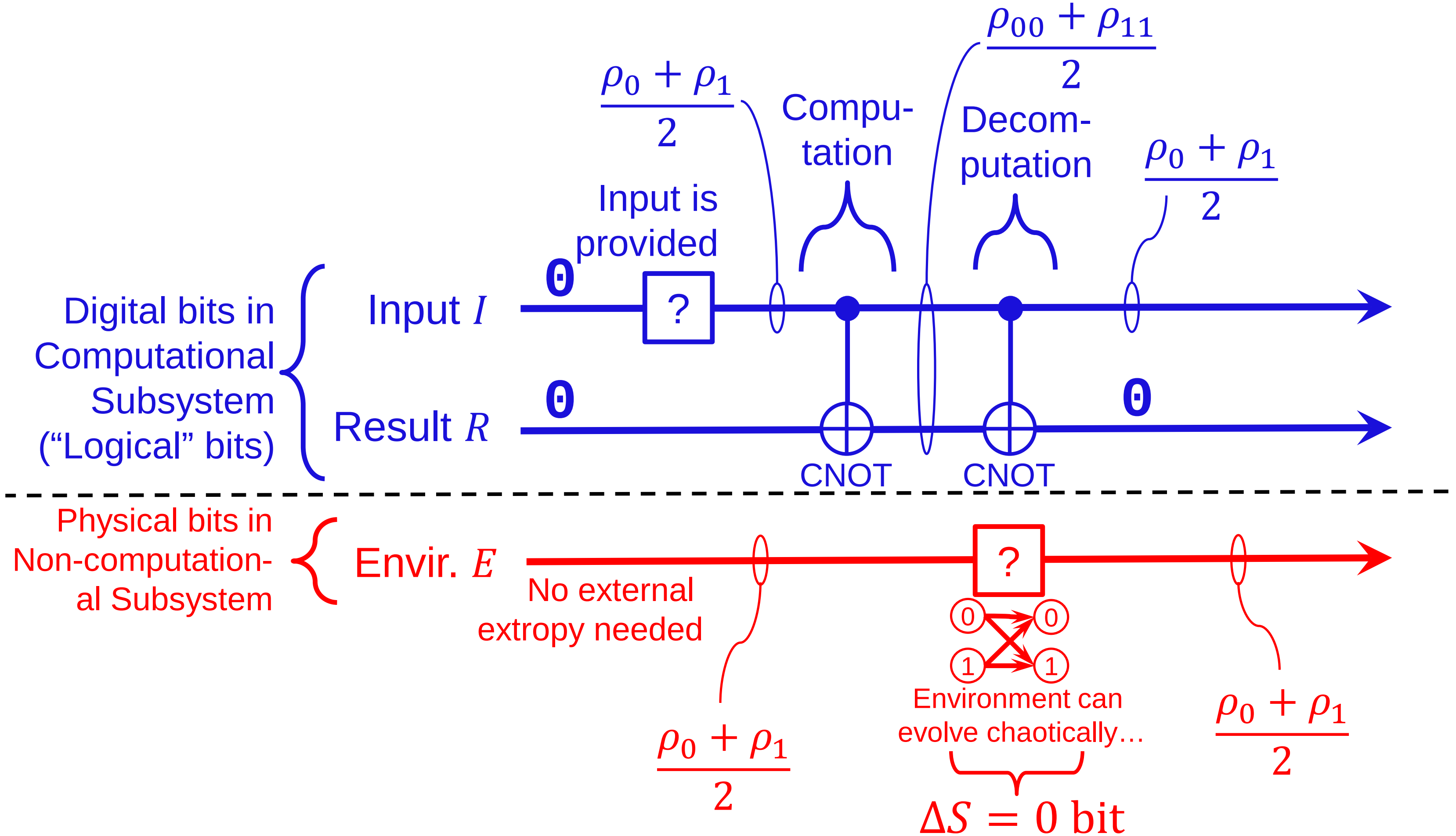


Logically Irreversible, Oblivious Erasure of a Correlated Bit



Moving a computed, correlated bit to an (*unpredictable!*) thermal environment *necessarily, inevitably* loses its correlations, and thus increases entropy!

Logically Reversible, Non-oblivious Decomputation of a Correlated Bit



Decomputing correlated bits, instead of ejecting them to the thermal environment, avoids losing correlations & increasing entropy!

This is why reversible computing (and **only** it!) can avoid Landauer’s limit…

Several recent empirical validations of Landauer's Principle

Redundant with the already very well-established, century-old basic facts of statistical physics, but hey, the doubters and skeptics are very stubborn people!

- Bérut *et al.*, 2012 (*Nature*)
  - Colloidal particle in a modulated double-well potential
  - Heat dissipation in bit erasure approached Landauer $kT \ln 2$ limit
- Orlov *et al.*, 2012 (*Japanese Journal of Applied Physics*)
  - Adiabatic charge transfer across a resistor
  - Verified that adiabatic transfers can dissipate $< kT \ln 2$
- Jun *et al.*, 2014 (*Physical Review Letters*)
  - Higher-precision version of Bérut experiment
  - Validated the limit, and that reversible transformations avoid it
- Yan *et al.*, 2018 (*Physical Review Letters*)
  - Quantum-mechanical experiment
  - Validated Landauer's Principle holds at single-atom level

Conclusion

- Landauer's principle really does follow directly as a simple and rigorous logical consequence of extremely fundamental insights of statistical physics that have been known for at least a century now…
  - The thermodynamic definitions of entropy and temperature
  - The reversibility of microphysics / second law of thermodynamics
  - Boltzmann & Planck's statistical understanding of physical entropy…
    - Von Neumann and Shannon only reformulated/reapplied it for specific domains!
      - Nothing fundamental about the Boltzmann/Planck definition was changed!
  - *Nothing else is needed* for the argument, except simple mathematics…
    - No other empirical inputs are even required to prove it!
    - We don't even need to make any equilibrium assumptions!
- However, to appreciate a number of subtleties of the proof is necessary if one wishes to help educate the skeptics… In particular:
  - The phenomenon of entropy intake in stochastic operations
  - The necessity of accounting for correlations
- In my opinion, the major barriers for our field are still *educational*:
  - Clarify these basic concepts → Engineering development will follow…