Explicit Partitioned Methods for Coupled Problems Based on Lagrange Multipliers

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Coupled Problems

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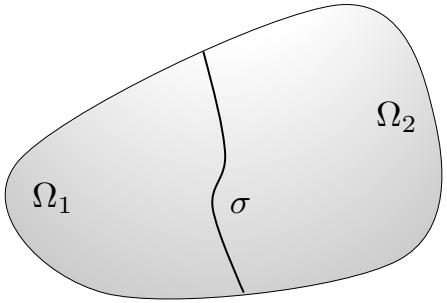
Partitioned Method for Interface Coupling

Objective:

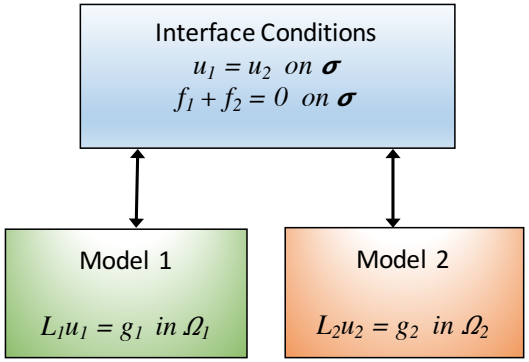
- Explicit method for time-dependent governing equations
- Each subdomain problem is solved independently
- Handles interfaces with non-matching grids
- Linearly consistent and second-order accurate
- Recovers single domain solution on matching grids

Two subdomains connected by an interface



Will describe two methods of coupling:

1. Variation Flux Recovery (VFR): Information exchange is limited to nodal masses and forces
2. Mortar Flux Recovery (MFR): Interface conditions are enforced through a flux boundary condition



Original Welded Interface Coupling

Governing Equations

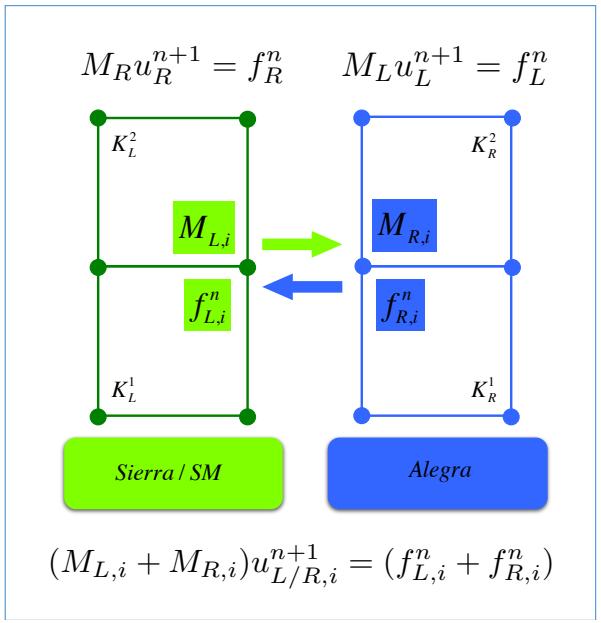
$$\rho_i \ddot{\mathbf{u}}_i - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_i) = f_i \quad \text{in } \Omega_i \times T$$
$$\mathbf{u}_i = g \quad \text{on } \Gamma_i \times T$$
$$\mathbf{u}_i(\mathbf{x}, 0) = \mathbf{u}_{i,0} \quad \text{in } \Omega_i$$
$$\dot{\mathbf{u}}_i(\mathbf{x}, 0) = \dot{\mathbf{u}}_{i,0} \quad \text{in } \Omega_i$$

$$\boldsymbol{\sigma} = \lambda(\nabla \cdot \mathbf{u}) + 2\mu\boldsymbol{\epsilon}(\mathbf{u}) \quad \boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

- Mass and forces swapped at nodes
- Minimally intrusive (black-box) coupling
- Requires matching interface nodes
- Uses lumped (diagonal) mass matrices
- Results in excessive mesh refinement when one code requires finer mesh

Interface Conditions

$$\boldsymbol{\sigma}_L \cdot \mathbf{n} = -\boldsymbol{\sigma}_R \cdot \mathbf{n} \quad \text{on } \sigma$$
$$\mathbf{u}_L = \mathbf{u}_R \quad \text{on } \sigma$$



Lagrange Multiplier Formulation

Monolithic formulation

$$\rho_1 \ddot{\mathbf{u}}_1 - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_1) = f_1 \quad \text{in } \Omega_1 \times T$$
$$\boldsymbol{\sigma}(\mathbf{u}_1) \cdot \mathbf{n} = \lambda \quad \text{on } \sigma \times T$$

$$\rho_2 \ddot{\mathbf{u}}_2 - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_2) = f_2 \quad \text{in } \Omega_2 \times T$$
$$\boldsymbol{\sigma}(\mathbf{u}_2) \cdot \mathbf{n} = \lambda \quad \text{on } \sigma \times T$$

$$\mathbf{u}_1 - \mathbf{u}_2 = 0 \quad \text{on } \sigma$$

$$\Longrightarrow$$

$$\mathbf{u}_1 \in V_1^h \subset H^1_{\Gamma_1}(\Omega_1)$$

$$\mathbf{u}_2 \in V_2^h \subset H^1_{\Gamma_2}(\Omega_2)$$

$$\lambda \in S^h \subset H^{-1/2}(\sigma)$$

Semi-discrete version

$$M_1 \ddot{\mathbf{u}}_1 + F_1(\mathbf{u}_1) = G_1^{\mathsf{T}} \lambda$$

$$M_2 \ddot{\mathbf{u}}_2 + F_2(\mathbf{u}_2) = -G_2^{\mathsf{T}} \lambda$$

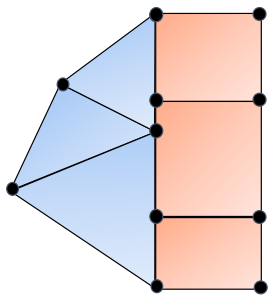
$$G_1 \mathbf{u}_1 - G_2 \mathbf{u}_2 = 0$$

- System of 3 equations for 3 unknowns
- Neumann boundary conditions involve unknown contact force λ
- Contact force continuity subsumed into the equations
- Displacement discontinuity enforced explicitly

Mass matrix: $M_{k,ij} = \int_\Omega \rho_k N_{k,i} N_{k,j} \, dS$

Coupling matrix: $G_{k,ij} = \int_\sigma N_{k,i} \hat{N}_j \, dS$

Subdomain basis functions $\{N_{k,i}\}$



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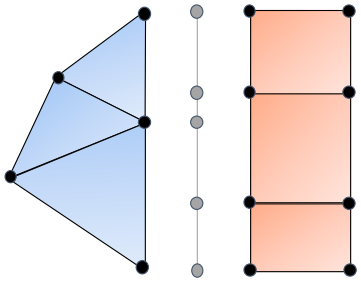
Lagrange multiplier basis functions $\{\hat{N}_i\}$



Lagrange Multiplier Formulation

Problem

- Lagrange multipliers not the most natural setting for partitioned schemes
- Index 2 Differential Algebraic Equation (DAE)
- Not compatible with explicit treatment of interface flux (λ)

Semi-discrete system

$$M_1 \ddot{\mathbf{u}}_1 + F_1(\mathbf{u}_1) = G_1^\mathsf{T} \lambda$$
$$M_2 \ddot{\mathbf{u}}_2 + F_2(\mathbf{u}_2) = -G_2^\mathsf{T} \lambda$$
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- Under suitable assumptions new constraint implies the original
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$$G_1\mathbf{u}_1 - G_2\mathbf{u}_2 = 0$$

$$\Downarrow$$

Modified semi-discrete system

$$M_1\ddot{\mathbf{u}}_1 + F_1(\mathbf{u}_1) = G_1^{\mathsf{T}}\lambda$$
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$$\left[\begin{array}{ccc|cc} M_{1,\sigma} & 0 & G_1^T & 0 & 0 \\ 0 & M_{2,\sigma} & -G_2^T & 0 & 0 \\ G_1 & -G_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & M_{1,0} & 0 \\ 0 & 0 & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \ddot{\boldsymbol{u}}_{2,\sigma} \\ \lambda \\ \hline \ddot{\boldsymbol{u}}_{1,0} \\ \ddot{\boldsymbol{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} f_{1,\sigma} \\ f_{2,\sigma} \\ 0 \\ \hline f_{1,0} \\ f_{2,0} \end{array}\right]$$

Semi-discrete system

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Lagrange Multiplier Formulation

Lagrange Multiplier Master Formulation

$$\left[\begin{array}{ccc|cc} M_{1,\sigma} & 0 & G_1^T & 0 & 0 \\ 0 & M_{2,\sigma} & -G_2^T & 0 & 0 \\ G_1 & -G_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & M_{1,0} & 0 \\ 0 & 0 & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \ddot{\boldsymbol{u}}_{2,\sigma} \\ \lambda \\ \hline \ddot{\boldsymbol{u}}_{1,0} \\ \ddot{\boldsymbol{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} f_{1,\sigma} \\ f_{2,\sigma} \\ 0 \\ \hline f_{1,0} \\ f_{2,0} \end{array}\right]$$

Option A (VFR)

$$\left[\begin{array}{c|cc} M_{1,\sigma} & 0 & G_1^T \\ \hline 0 & M_{2,\sigma} & -\tilde{M}_{2,\sigma} \\ G_1 & -\tilde{M}_{2,\sigma} & 0 \end{array}\right] \left[\begin{array}{c} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \hline \ddot{\boldsymbol{u}}_{2,\sigma} \\ \lambda \end{array}\right] = \left[\begin{array}{c} f_{1,\sigma} \\ \hline f_{2,\sigma} \\ 0 \end{array}\right]$$

$$\left[\begin{array}{c|cc} M_{2,\sigma} & 0 & -G_2^T \\ \hline 0 & M_{1,\sigma} & \tilde{M}_{1,\sigma} \\ G_2 & -\tilde{M}_{1,\sigma} & 0 \end{array}\right] \left[\begin{array}{c} \ddot{\boldsymbol{u}}_{2,\sigma} \\ \hline \ddot{\boldsymbol{u}}_{1,\sigma} \\ \lambda \end{array}\right] = \left[\begin{array}{c} f_{2,\sigma} \\ \hline f_{1,\sigma} \\ 0 \end{array}\right]$$

Option B (MFR)

$$\left[\begin{array}{cc|c} M_{1,\sigma} & 0 & G_1^T \\ 0 & M_{2,\sigma} & -G_2^T \\ \hline G_1 & -G_2 & 0 \end{array}\right] \left[\begin{array}{c} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \ddot{\boldsymbol{u}}_{2,\sigma} \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} f_{1,\sigma} \\ f_{2,\sigma} \\ \hline 0 \end{array}\right]$$

$$(G_1 M_{1,\sigma}^{-1} G_1^T + G_2 M_{2,\sigma}^{-1} G_2^T)\lambda = G_2 M_{2,\sigma}^{-1} f_{2,\sigma} - G_1 M_{1,\sigma}^{-1} f_{1,\sigma}$$

- Requires 2 separate master systems for each side
- λ collocated with displacement on the opposite side
- λ spaces simple to construct
- Results in a generalized mass-force exchange

- Leads to a system for interface DOFs
- Requires common mesh refinement
- Requires preconditioning, but potentially more accuarate
- Related to FETI, hybrid methods

Formulation on Matching Grids

$$\begin{bmatrix} M_{1,\sigma} & 0 & G_1^T & 0 & 0 \\ 0 & M_{2,\sigma} & -G_2^T & 0 & 0 \\ G_1 & -G_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{1,0} & 0 \\ 0 & 0 & 0 & 0 & M_{2,0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \ddot{\boldsymbol{u}}_{2,\sigma} \\ \lambda \\ \ddot{\boldsymbol{u}}_{1,0} \\ \ddot{\boldsymbol{u}}_{2,0} \end{bmatrix} = \begin{bmatrix} f_{1,\sigma} \\ f_{2,\sigma} \\ 0 \\ f_{1,0} \\ f_{2,0} \end{bmatrix}$$

$$\begin{bmatrix} M_{1,\sigma} & 0 & \tilde{M}_{1,\sigma} & 0 & 0 \\ 0 & M_{2,\sigma} & -\tilde{M}_{2,\sigma} & 0 & 0 \\ \tilde{M}_{1,\sigma} & -\tilde{M}_{2,\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{1,0} & 0 \\ 0 & 0 & 0 & 0 & M_{2,0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \ddot{\boldsymbol{u}}_{2,\sigma} \\ \lambda \\ \ddot{\boldsymbol{u}}_{1,0} \\ \ddot{\boldsymbol{u}}_{2,0} \end{bmatrix} = \begin{bmatrix} f_{1,\sigma} \\ f_{2,\sigma} \\ 0 \\ f_{1,0} \\ f_{2,0} \end{bmatrix}$$

Matching nodes specialization

Option A: (1,1) Schur

eliminate **interface** DOF

$$\begin{bmatrix} M_{1,\sigma} & 0 & \tilde{M}_{1,\sigma} \\ 0 & M_{2,\sigma} & -\tilde{M}_{2,\sigma} \\ \tilde{M}_{1,\sigma} & -\tilde{M}_{2,\sigma} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \ddot{\boldsymbol{u}}_{2,\sigma} \\ \lambda \end{bmatrix} = \begin{bmatrix} f_{1,\sigma} \\ f_{2,\sigma} \\ 0 \end{bmatrix}$$

Option B: (2,2) Schur

eliminate **internal** DOF

$$\begin{bmatrix} M_{1,\sigma} & 0 & \tilde{M}_{1,\sigma} \\ 0 & M_{2,\sigma} & -\tilde{M}_{2,\sigma} \\ \tilde{M}_{1,\sigma} & -\tilde{M}_{2,\sigma} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \ddot{\boldsymbol{u}}_{2,\sigma} \\ \lambda \end{bmatrix} = \begin{bmatrix} f_{1,\sigma} \\ f_{2,\sigma} \\ 0 \end{bmatrix}$$

$$\tilde{M}_{*,\sigma}(M_{1,\sigma}^{-1} + M_{2,\sigma}^{-1})\tilde{M}_{*,\sigma}\lambda = \tilde{M}_{*,\sigma}(M_{2,\sigma}^{-1} f_{2,\sigma} - M_{1,\sigma}^{-1} f_{1,\sigma})$$

VFR

MFR

$$\begin{aligned} (M_{1,\sigma}^i + M_{2,\sigma}^i)\ddot{\boldsymbol{u}}_{1,\sigma}^i &= f_{1,\sigma}^i + f_{2,\sigma}^i \\ (M_{1,\sigma}^i + M_{2,\sigma}^i)\ddot{\boldsymbol{u}}_{2,\sigma}^i &= f_{1,\sigma}^i + f_{2,\sigma}^i \end{aligned}$$

Identical Partitioned systems for matching nodes!

VFR Formulation for Non-Matching Grids

$$\left[\begin{array}{ccc|cc} M_{1,\sigma} & 0 & G_1^T & 0 & 0 \\ 0 & M_{2,\sigma} & -G_2^T & 0 & 0 \\ G_1 & -G_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & M_{1,0} & 0 \\ 0 & 0 & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\boldsymbol{u}}_{1,\sigma} \\ \ddot{\boldsymbol{u}}_{2,\sigma} \\ \lambda \\ \hline \ddot{\boldsymbol{u}}_{1,0} \\ \ddot{\boldsymbol{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} f_{1,\sigma} \\ f_{2,\sigma} \\ 0 \\ \hline f_{1,0} \\ f_{2,0} \end{array}\right]$$

Consider system for subdomain 1 where:

$$G_{1,ij} = \int_{\Omega_1} N_{1,i} N_{2,j} d\Omega \quad G_{2,ij} = \int_{\Omega_1} N_{2,i} N_{2,j} d\Omega = M_{2,\sigma}$$

Take the (1,1) Schur complement and let $G_1 \approx M_{1,\sigma} \overleftarrow{P}$ *:*

$$(M_1 + M_{1\sigma} \overleftarrow{P} M_{2\sigma}^{-1} \overrightarrow{P}) \ddot{u}_1 = F_1 + M_{1\sigma} \overleftarrow{P} M_{2\sigma}^{-1} F_2$$

Similarly:

$$(M_2 + M_{2\sigma} \overleftarrow{P} M_{1\sigma}^{-1} \overrightarrow{P}) \ddot{u}_2 = F_2 + M_{1\sigma} \overleftarrow{P} M_{1\sigma}^{-1} F_1$$

Generalization of Mass-Force exchange

VFR Equivalence to Single Domain Solution with Matching Nodes

For interfaces with matching nodes there holds $M_{1\sigma} = M_{1\sigma}$ *and* $\overleftarrow{P} = \overrightarrow{P} = \mathbf{I}$

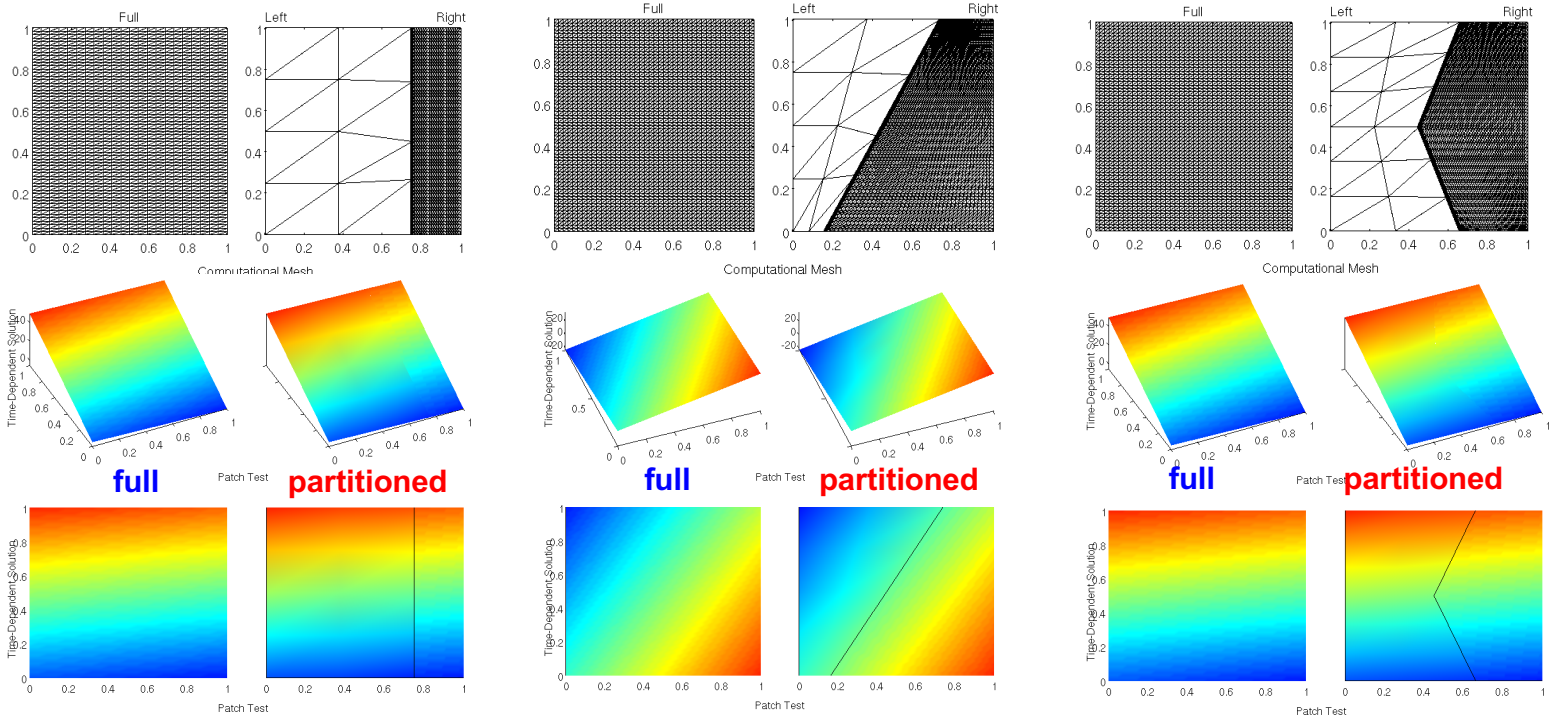
$$\left(M_1 + M_{1\sigma}\overleftarrow{P}M_{2\sigma}^{-1}M_2\overrightarrow{P}\right)\ddot{u}_1 = F_1 + M_{1\sigma}\overleftarrow{P}M_{2\sigma}^{-1}F_2 \qquad \left(M_2 + M_{2\sigma}\overrightarrow{P}M_{1\sigma}^{-1}M_1\overleftarrow{P}\right)\ddot{u}_2 = F_2 + M_{2\sigma}\overrightarrow{P}M_{1\sigma}^{-1}F_1$$

$$\left(M_1 + M_2\right)\ddot{u}_1 = F_1 + F_2 \qquad \left(M_2 + M_1\right)\ddot{u}_2 = F_2 + F_1$$

Interface	Vertical	Slanted
Mesh Ω_1	24x20	24x20
Mesh Ω_2	24x20	24x20
L_2 error Ω_1	3.38E-17	9.43E-17
L_2 error Ω_2	1.07E-15	1.05E-15
L_2 error Ω_1	2.49E-15	7.74E-15
L_2 error Ω_2	9.92E-14	1.23E-13

Patch Test on Non-Matching Interfaces

Recovers linear displacements

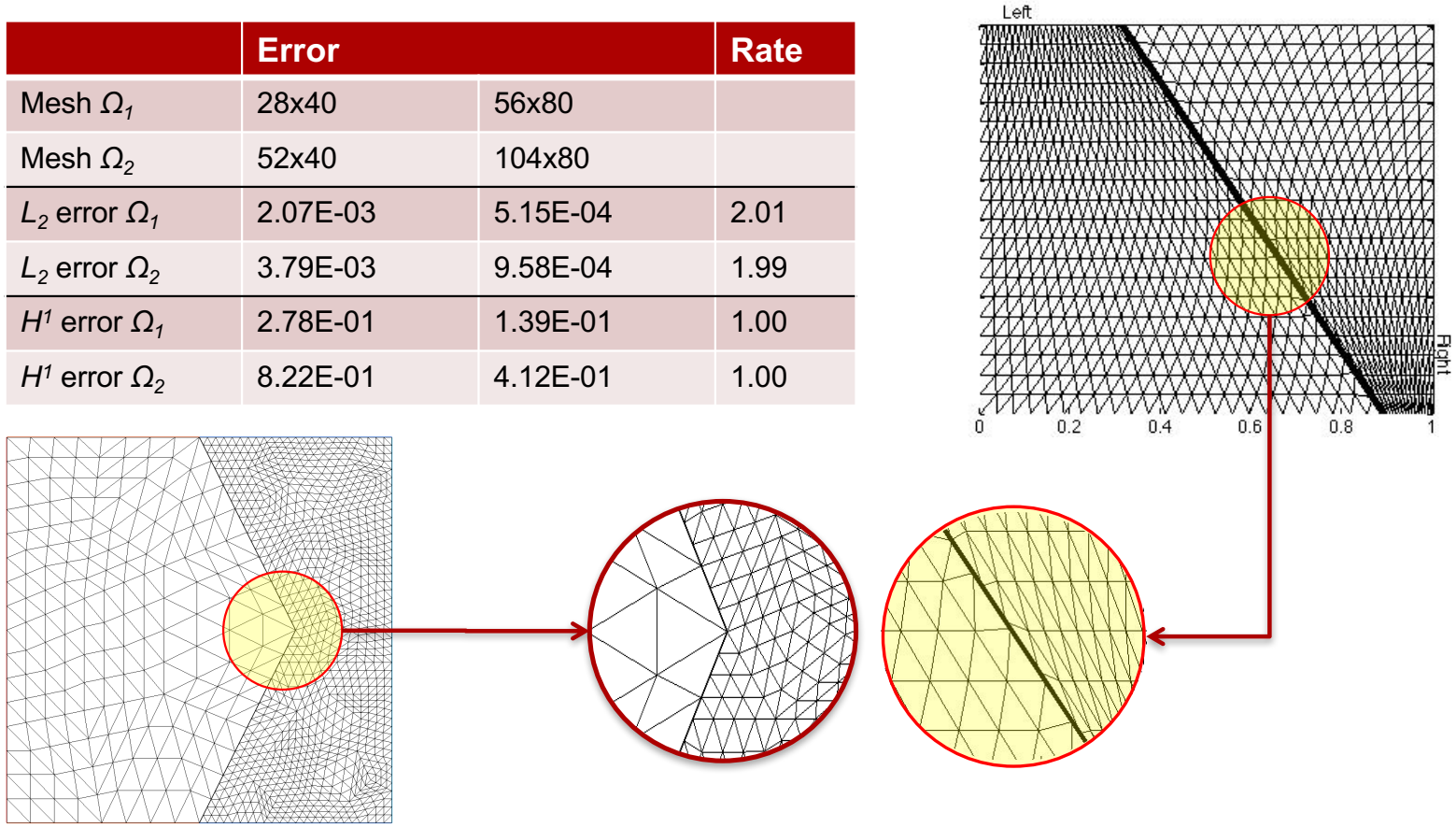


Convergence Results

Recovers linear displacements

2nd order accuracy on non-matching interfaces

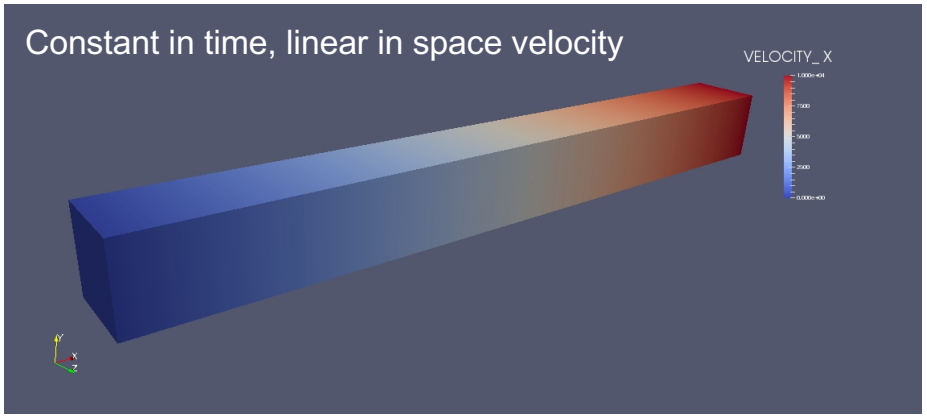
	Error		Rate
Mesh Ω_1	28x40	56x80	
Mesh Ω_2	52x40	104x80	
L_2 error Ω_1	2.07E-03	5.15E-04	2.01
L_2 error Ω_2	3.79E-03	9.58E-04	1.99
H^1 error Ω_1	2.78E-01	1.39E-01	1.00
H^1 error Ω_2	8.22E-01	4.12E-01	1.00



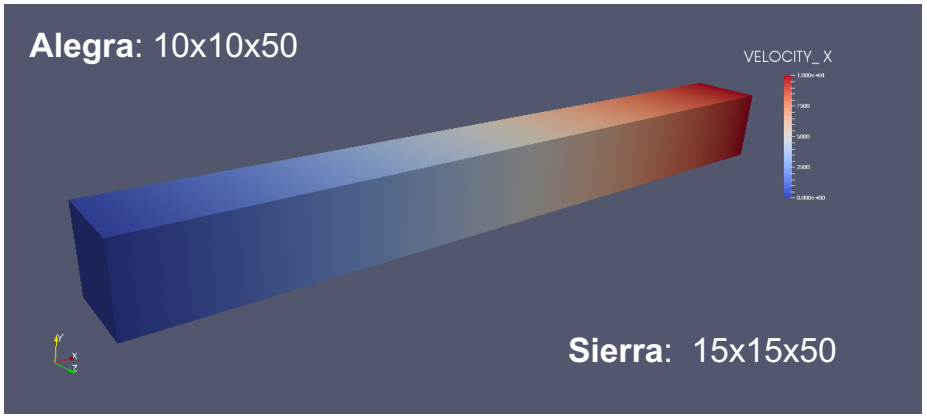
Consistency Test in Application Codes

Consistency test

Exact



Forte+VFR



Lagrange Multiplier Formulation

Consider formulation for scalar conservation equation

Monolithic formulation

$$\dot{\varphi}_1 - \nabla \cdot f_1(\varphi_1) = g_1 \quad \text{in } \Omega_1 \times T$$
$$f_1(\varphi_1) \cdot \mathbf{n} = \lambda \quad \text{on } \sigma \times T$$

$$\dot{\varphi}_2 - \nabla \cdot f_2(\varphi_2) = g_2 \quad \text{in } \Omega_2 \times T$$
$$f_2(\varphi_2) \cdot \mathbf{n} = -\lambda \quad \text{on } \sigma \times T$$

$$\varphi_1 - \varphi_2 = 0 \quad \text{on } \sigma$$

$$\Longrightarrow$$

$$\varphi_1 \in V_1^h \subset H^1_{\Gamma_1}(\Omega_1)$$

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$$\lambda \in S^h \subset H^{-1/2}(\sigma)$$

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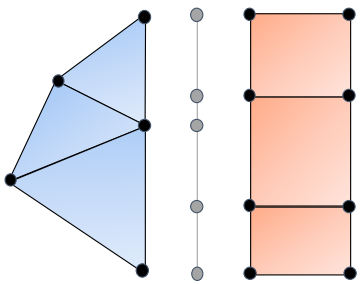

Mortar Flux Recovery (MFR)

- As before for VFR, replace the original constraint with $G_1\dot{\varphi}_1 - G_2\dot{\varphi}_2 = 0$
- With suitable assumptions the two constraints are qequivalent
- New system enables a fully explicit treatment of λ

Modified semi-discrete system

$$M_1\dot{\varphi}_1 + f_1(\varphi_1) = G_1^\mathsf{T}\lambda$$
$$M_2\dot{\varphi}_2 + f_2(\varphi_2) = -G_2^\mathsf{T}\lambda$$
$$G_1\dot{\varphi}_1 - G_2\dot{\varphi}_2 = 0$$

Expanded semi-discrete linear system

$$\begin{pmatrix} M_{1,II} & M_{1,I\sigma} & 0 & 0 & 0 \\ M_{1,\sigma I} & M_{1,\sigma\sigma} & 0 & 0 & G_1^\mathsf{T} \\ 0 & 0 & M_{2,II} & M_{2,I\sigma} & 0 \\ 0 & 0 & M_{2,\sigma I} & M_{2,\sigma\sigma} & -G_2^\mathsf{T} \\ 0 & G_1 & 0 & -G_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{\varphi}_{1,I} \\ \dot{\varphi}_{1,\sigma} \\ \dot{\varphi}_{2,I} \\ \dot{\varphi}_{2,\sigma} \\ \lambda \end{pmatrix} = \begin{pmatrix} F_{1,I}(\varphi_1) \\ F_{1,\sigma}(\varphi_1) \\ F_{2,I}(\varphi_2) \\ F_{2,\sigma}(\varphi_2) \\ 0 \end{pmatrix}$$

Interface linear system

$$\begin{pmatrix} A_1 & 0 & G_1^\mathsf{T} \\ 0 & A_2 & -G_2^\mathsf{T} \\ G_1 & -G_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{\varphi}_{1,\sigma} \\ \dot{\varphi}_{2,\sigma} \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{f}_1(\varphi_1) \\ \tilde{f}_2(\varphi_2) \\ 0 \end{pmatrix}$$

$$A_1 = M_{1,\sigma\sigma} - M_{1,\sigma I} M_{1,II}^{-1} M_{1,I\sigma}$$
$$A_2 = M_{2,\sigma\sigma} - M_{2,\sigma I} M_{2,II}^{-1} M_{2,I\sigma}$$
$$\tilde{f}_1 = f_{1,\sigma} - M_{1,\sigma I} M_{1,II}^{-1} f_{1,I}$$
$$\tilde{f}_2 = f_{2,\sigma} - M_{2,\sigma I} M_{2,II}^{-1} f_{2,I}$$

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- New system enables a fully explicit treatment of λ

Modified semi-discrete system

$$M_1\dot{\varphi}_1 + f_1(\varphi_1) = G_1^\mathsf{T}\lambda$$
$$M_2\dot{\varphi}_2 + f_2(\varphi_2) = -G_2^\mathsf{T}\lambda$$
$$G_1\dot{\varphi}_1 - G_2\dot{\varphi}_2 = 0$$

Expanded semi-discrete linear system

$$\begin{pmatrix} M_{1,II} & M_{1,I\sigma} & 0 & 0 & 0 \\ M_{1,\sigma I} & M_{1,\sigma\sigma} & 0 & 0 & G_1^\mathsf{T} \\ 0 & 0 & M_{2,II} & M_{2,I\sigma} & 0 \\ 0 & 0 & M_{2,\sigma I} & M_{2,\sigma\sigma} & -G_2^\mathsf{T} \\ 0 & G_1 & 0 & -G_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{\varphi}_{1,I} \\ \dot{\varphi}_{1,\sigma} \\ \dot{\varphi}_{2,I} \\ \dot{\varphi}_{2,\sigma} \\ \lambda \end{pmatrix} = \begin{pmatrix} F_{1,I}(\varphi_1) \\ F_{1,\sigma}(\varphi_1) \\ F_{2,I}(\varphi_2) \\ F_{2,\sigma}(\varphi_2) \\ 0 \end{pmatrix}$$

Interface linear system

$$\begin{pmatrix} A_1 & 0 & G_1^\mathsf{T} \\ 0 & A_2 & -G_2^\mathsf{T} \\ G_1 & -G_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{\varphi}_{1,\sigma} \\ \dot{\varphi}_{2,\sigma} \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{f}_1(\varphi_1) \\ \tilde{f}_2(\varphi_2) \\ 0 \end{pmatrix}$$

$$A_1 = M_{1,\sigma\sigma} - M_{1,\sigma I} M_{1,II}^{-1} M_{1,I\sigma}$$
$$A_2 = M_{2,\sigma\sigma} - M_{2,\sigma I} M_{2,II}^{-1} M_{2,I\sigma}$$
$$\tilde{f}_1 = f_{1,\sigma} - M_{1,\sigma I} M_{1,II}^{-1} f_{1,I}$$
$$\tilde{f}_2 = f_{2,\sigma} - M_{2,\sigma I} M_{2,II}^{-1} f_{2,I}$$

$$\lambda(\varphi_1, \varphi_2) = \left(G_1 A_1^{-1} G_1^\mathsf{T} + G_2 A_2^{-1} G_2^\mathsf{T}\right)^{-1} \left(G_1 A_1^{-1} \tilde{f}_1(\varphi_1) - G_2 A_2^{-1} \tilde{f}_2(\varphi_2)\right)$$

Mortar Flux Recovery (MFR)

MFR algorithm

1. Solve for λ

$$\lambda(\varphi_1^n, \varphi_2^n) = \left(G_1 A_1^{-1} G_1^\mathsf{T} + G_2 A_2^{-1} G_2^\mathsf{T}\right)^{-1} \left(G_1 A_1^{-1} \tilde{f}_1(\varphi_1^n) - G_2 A_2^{-1} \tilde{f}_2(\varphi_2^n)\right)$$

2. Solve each subdomain equation independently

$$M_1(\varphi_1^{n+1} - \varphi_1^n) = \Delta t \left(f_1(\varphi_1^n) + G_1^\mathsf{T} \lambda(\varphi_1^n, \varphi_2^n)\right)$$
$$M_2(\varphi_2^{n+1} - \varphi_2^n) = \Delta t \left(f_2(\varphi_2^n) - G_2^\mathsf{T} \lambda(\varphi_1^n, \varphi_2^n)\right)$$

- MFR is an explicit method to estimate Neumann flux on interface boundary
- Equation for λ is similar to interface equation in FETI method*
  - domain decomposition method with goal to increase concurrency
  - developed originally for elliptic problems
  - for time dependent problems, discretize in time then solve the monolithic problem implicitly

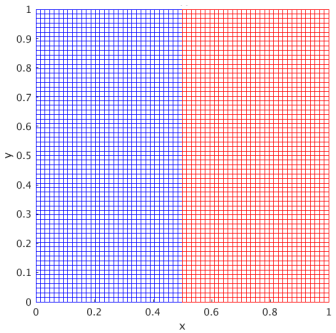
* C. Farhat, F. Roux, A method of finite element tearing and interconnecting and its parallel solution algorithm, IJNME 32, 1991
* C. Farhat, L. Crivelli, F. Roux, A transient FETI methodology for large-scale parallel implicit computations in structural mechanics, IJNME 37, 1994
* A. Toselli, FETI domain decomposition methods for scalar advection-diffusion problems, CMAME 190, 2001

Numerical Results: Manufactured Solution

Matching grid comparison to single domain

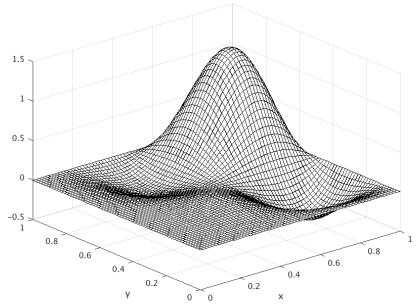
$$\varphi_k(\mathbf{x}, t) = x^2 y \sin(2\pi x) \sin(2\pi y) \exp(t)$$

Mesh



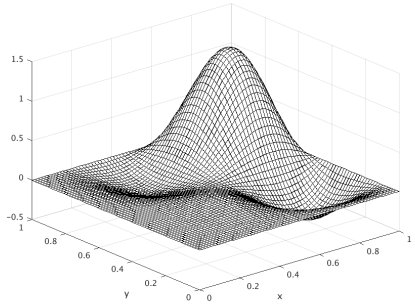
MFR

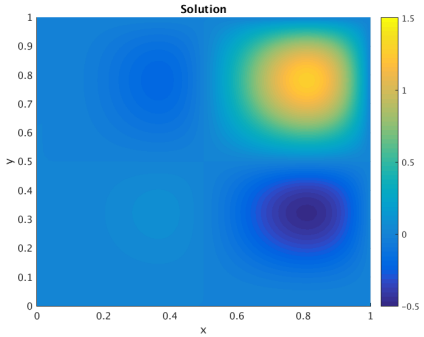
advection dominated



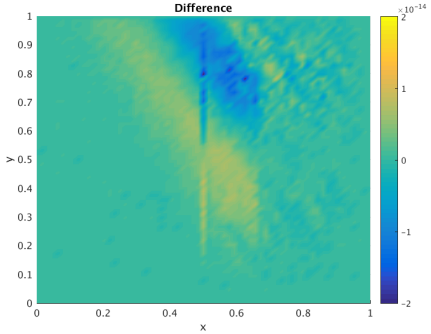
MFR

pure diffusion

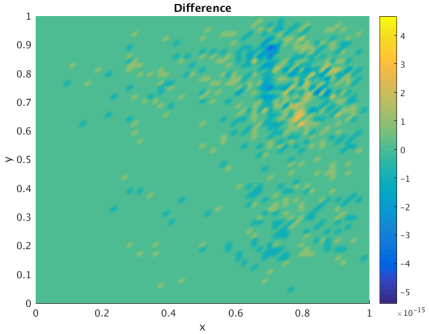




$(-0.5, 1.5)$



$(-2 \times 10^{-14}, 2 \times 10^{-14})$



$(-5 \times 10^{-15}, 5 \times 10^{-15})$

Numerical Results: Patch Test

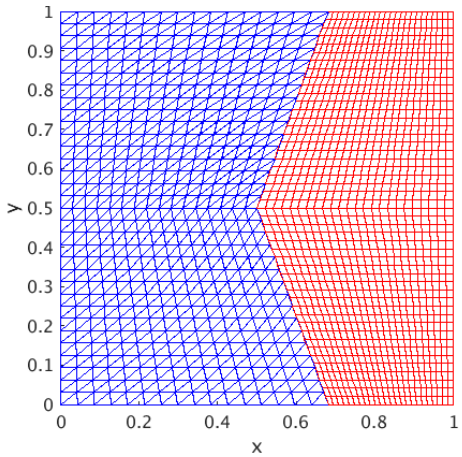
Manufactured solution: $\varphi_k(\mathbf{x}, t) = x + y$

$$\dot{\varphi}_k - \nabla \cdot (\epsilon \nabla \varphi_k - \mathbf{v}\varphi_k) = g_k$$

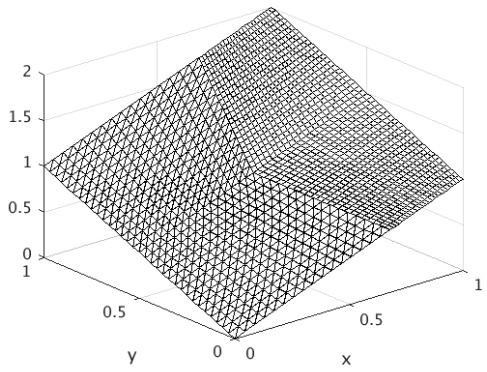
with $\epsilon = 0.001$, $\mathbf{v} = (-\sin(\pi/6), \cos(\pi/6))$

Stabilize with SUPG

Mesh



MFR $t = 1$

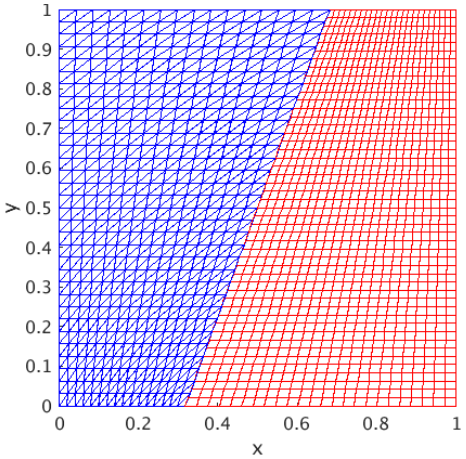


Recovers linear manufactured solution to machine precision

Numerical Results: Convergence

$$\varphi_k(\mathbf{x}, t) = x^2 y \sin(2\pi x) \sin(2\pi y) \exp(t)$$

Mesh



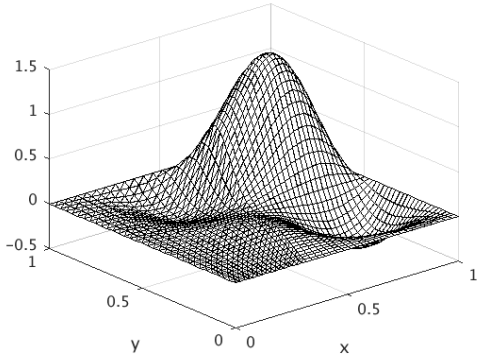
$$\dot{\varphi}_k - \nabla \cdot (\epsilon \nabla \varphi_k - \mathbf{v} \varphi_k) = g_k$$

$$\mathbf{v} = \left(-\sin(\tfrac{\pi}{6}), \cos(\tfrac{\pi}{6}) \right)$$

$$\epsilon = 0.001$$

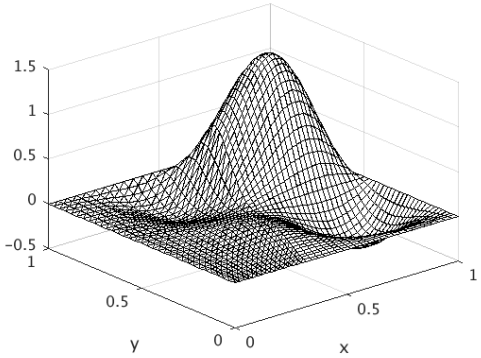
MFR $t = 1$

advection dominated

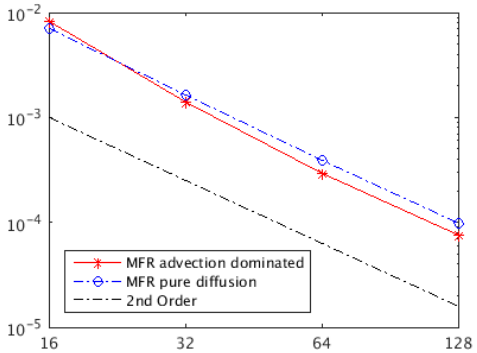


MFR $t = 1$

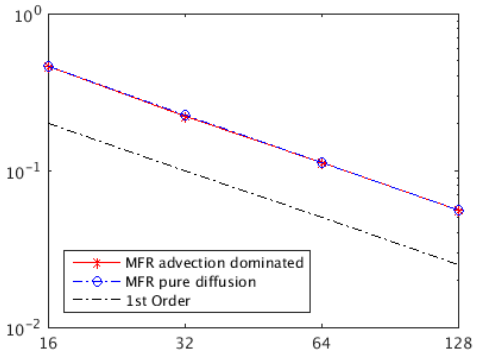
pure diffusion$t = 1$



L^2 Error



H^1 Error

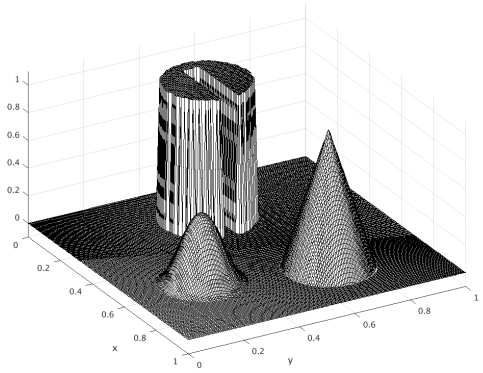


Numerical Results: Pure Advection

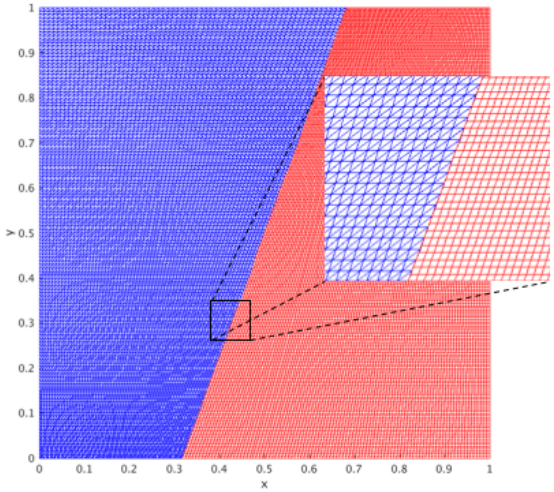
Rotating velocity field: $\mathbf{v} = (-y + \frac{1}{2}, x - \frac{1}{2})$

$$\dot{\varphi}_k + \nabla \cdot (\mathbf{v}\varphi_k) = 0$$

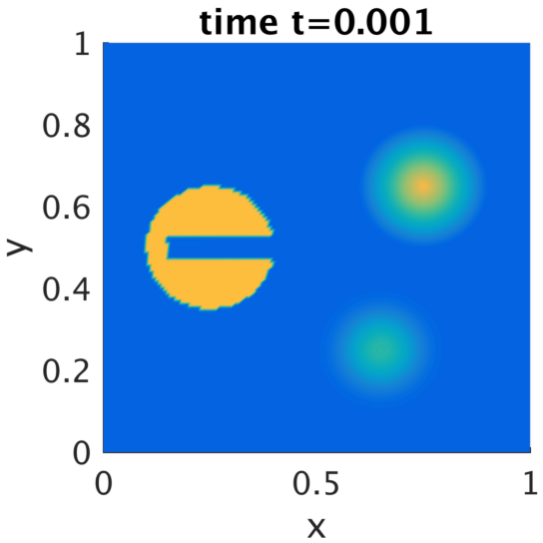
SUPG stabilization



Mesh



MFR



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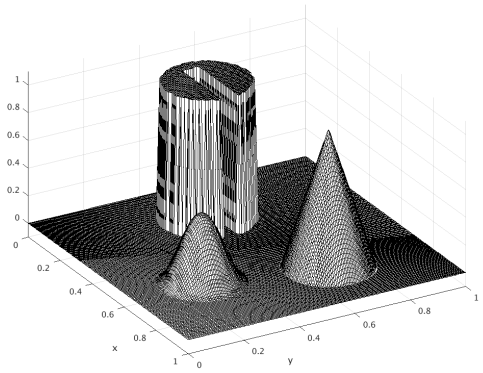

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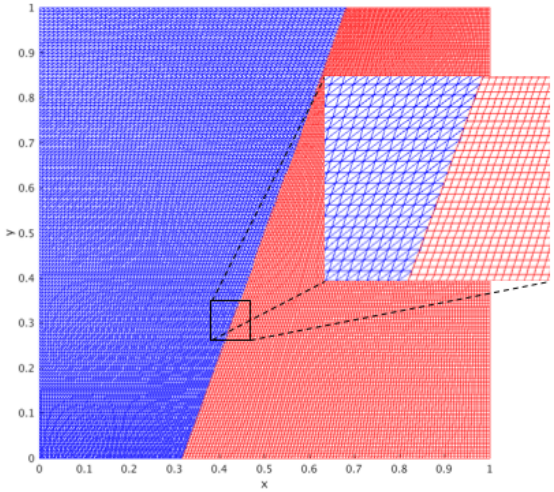

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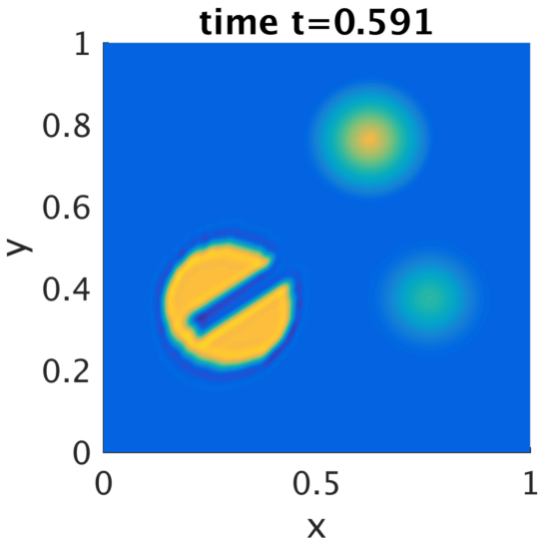

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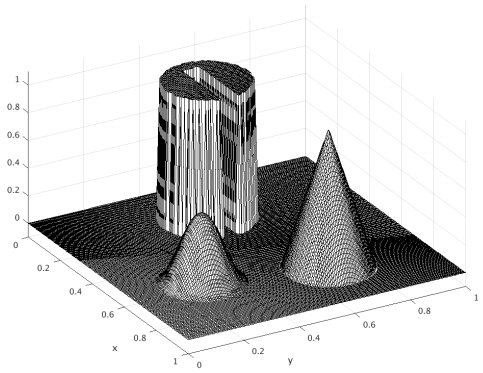

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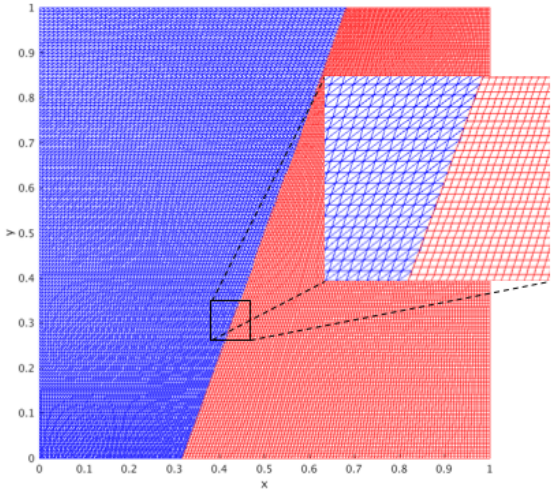

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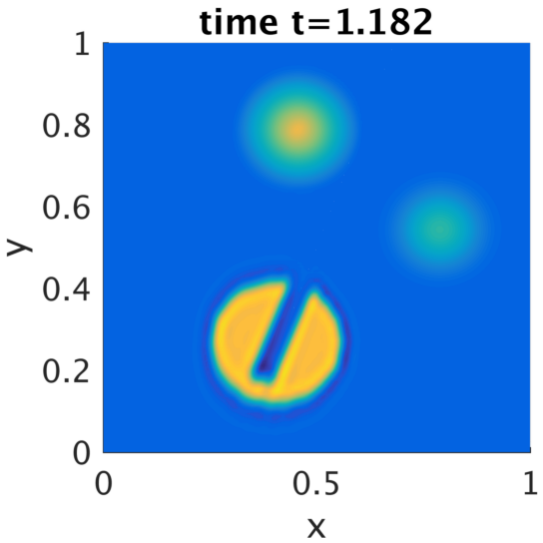

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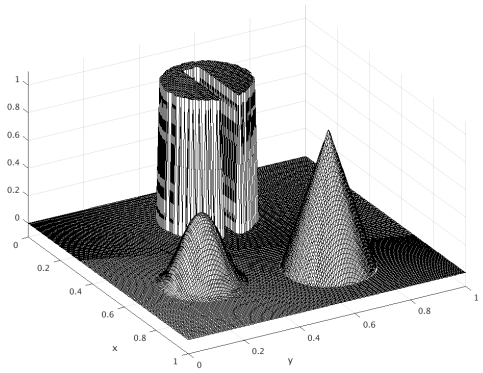
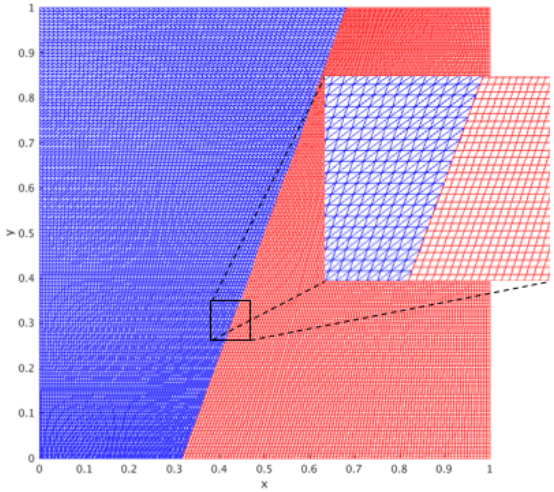


Mesh



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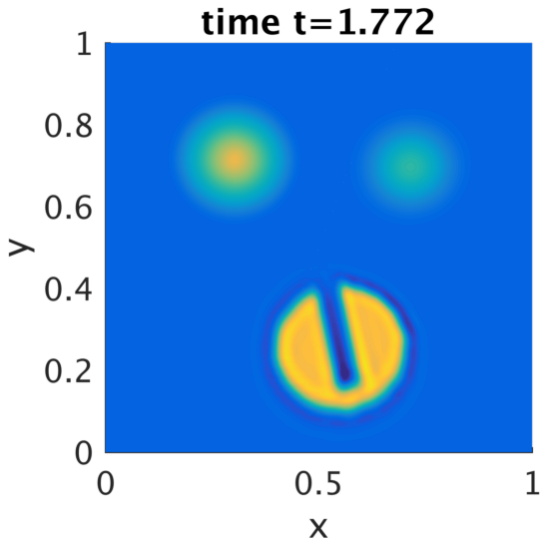

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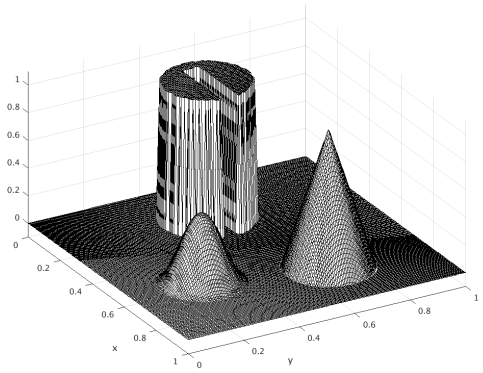

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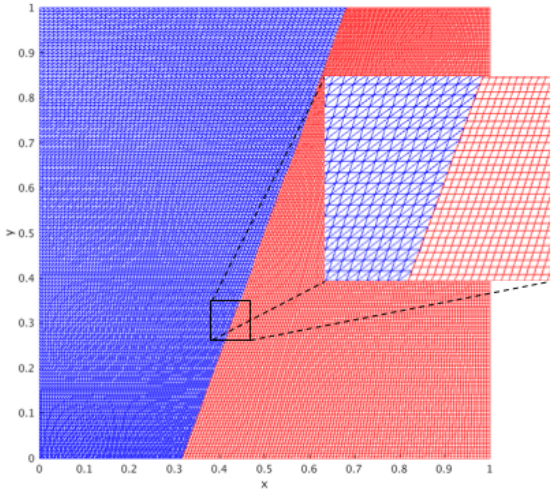

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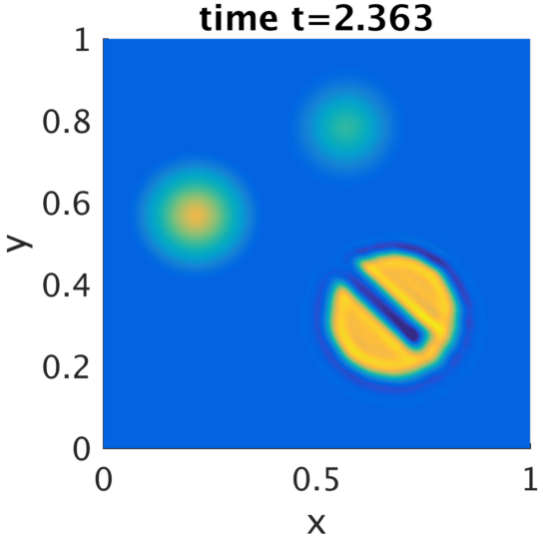

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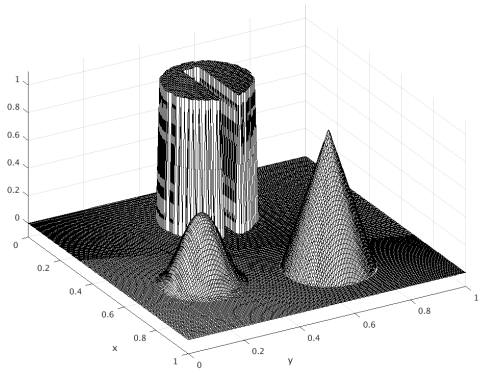

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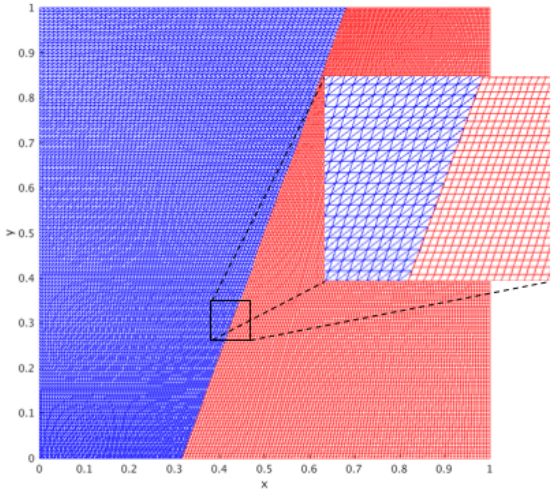

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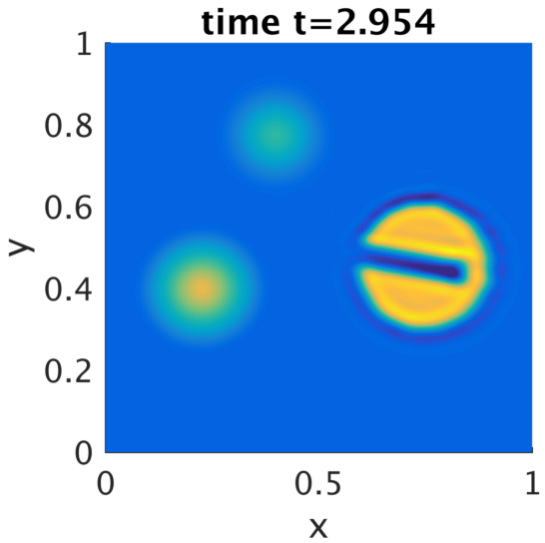

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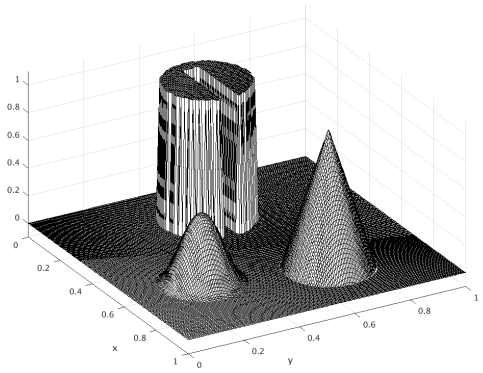

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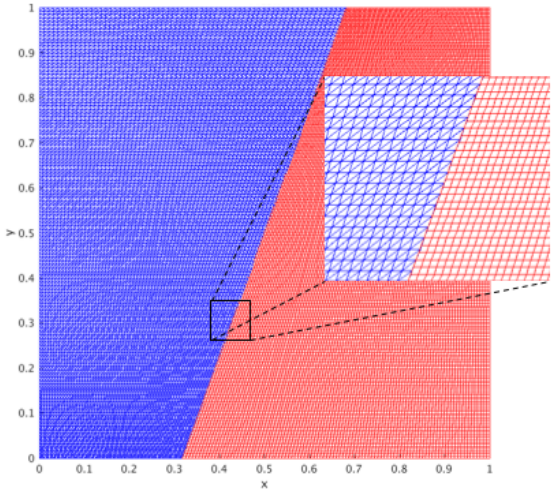

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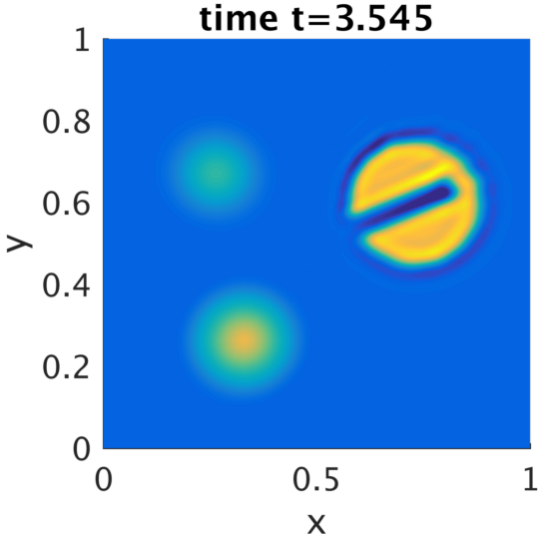

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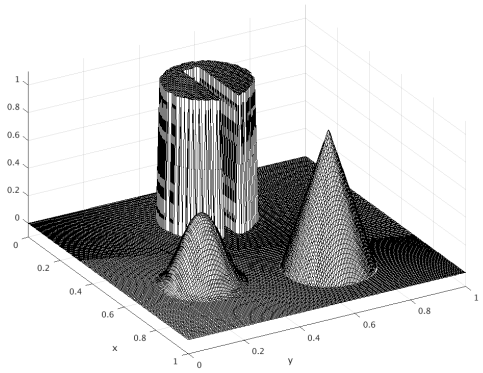

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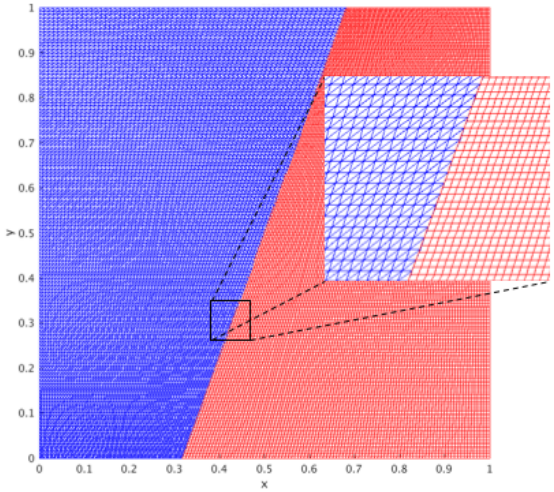

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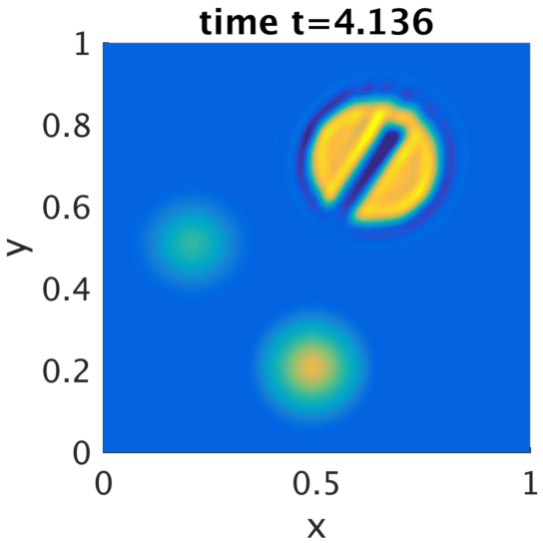

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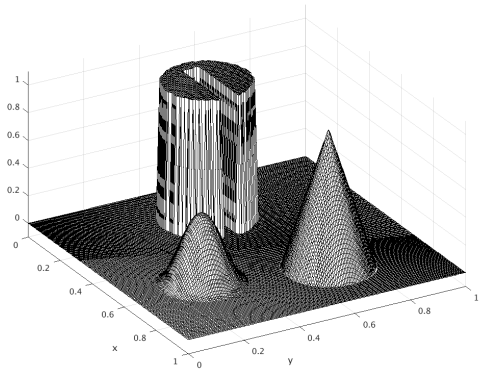

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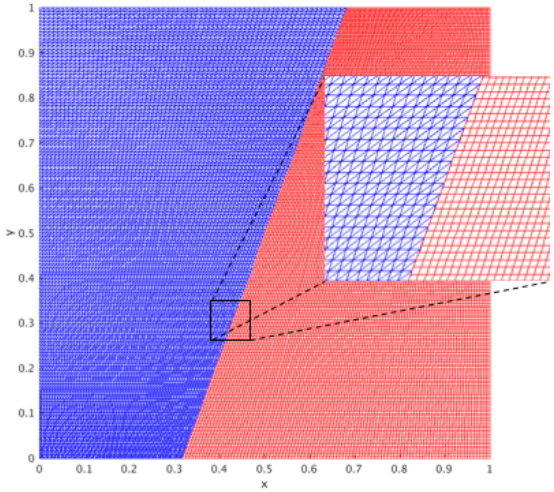

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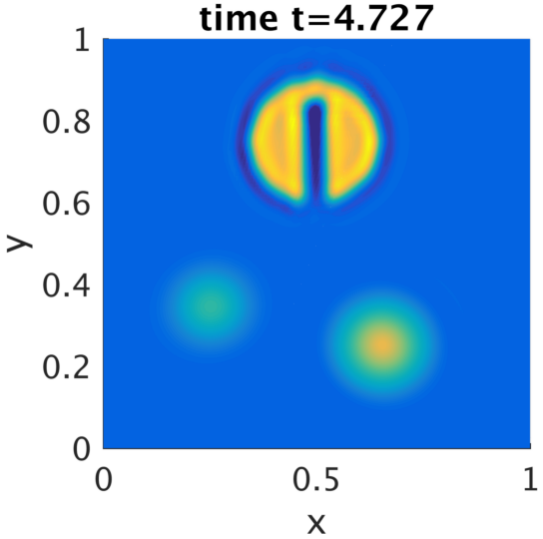

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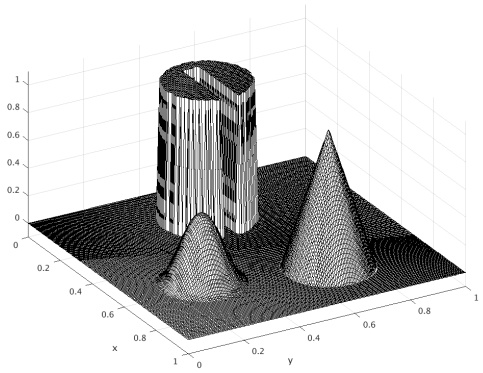
SUPG stabilization



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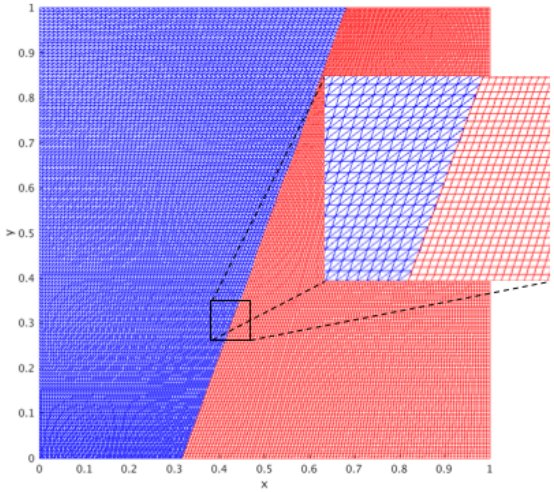

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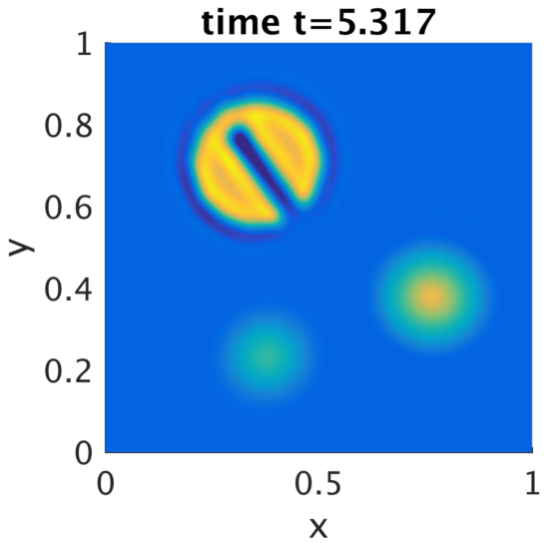

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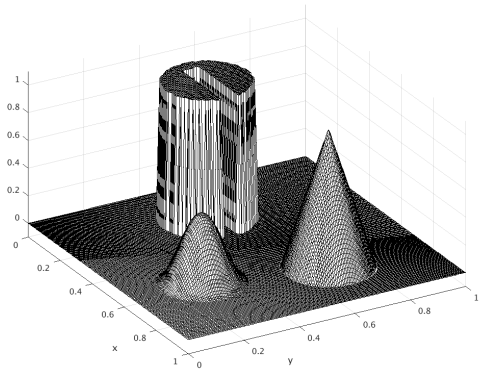
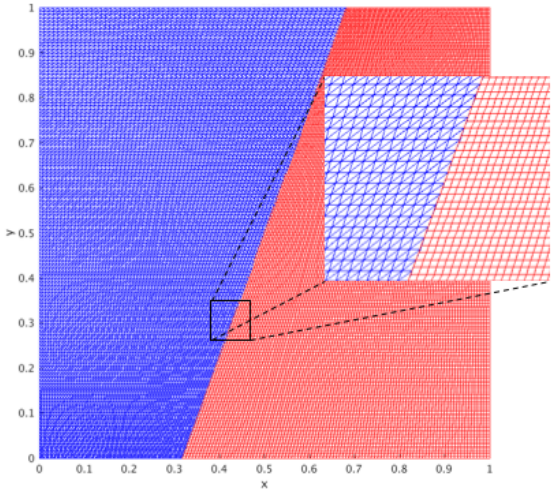
$$\dot{\varphi}_k + \nabla \cdot (\mathbf{v}\varphi_k) = 0$$

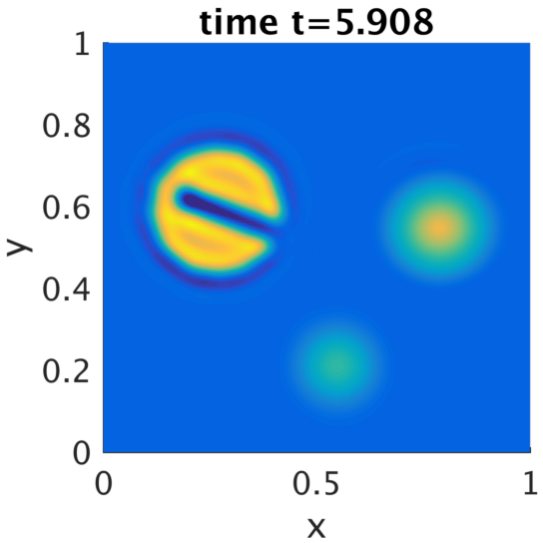
SUPG stabilization



Mesh



MFR

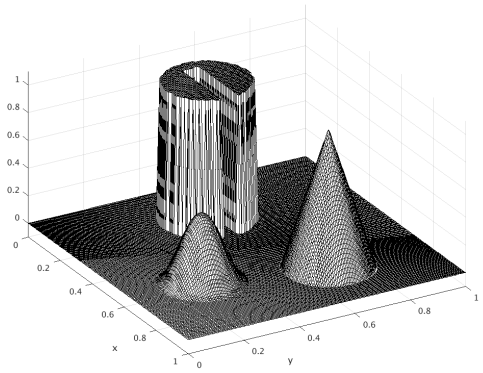


Numerical Results: Pure Advection

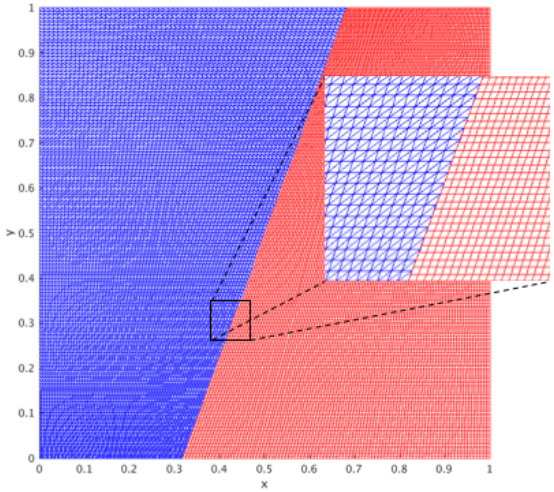
Rotating velocity field: $\mathbf{v} = (-y + \frac{1}{2}, x - \frac{1}{2})$

$$\dot{\varphi}_k + \nabla \cdot (\mathbf{v}\varphi_k) = 0$$

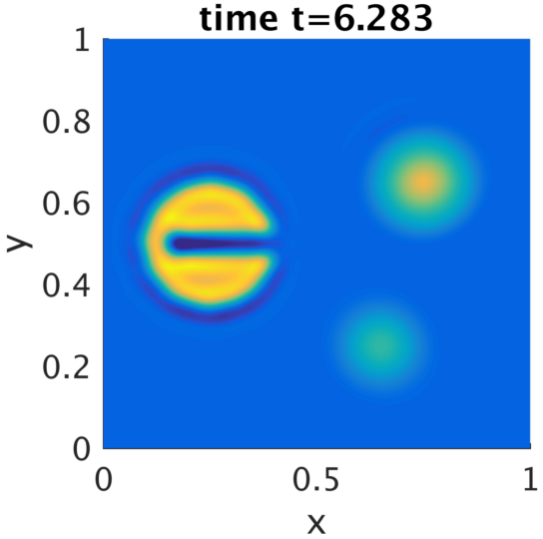
SUPG stabilization



Mesh



MFR



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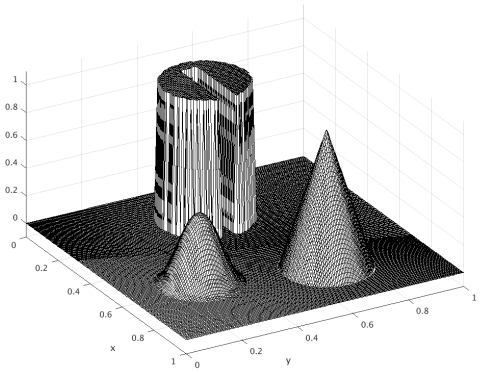
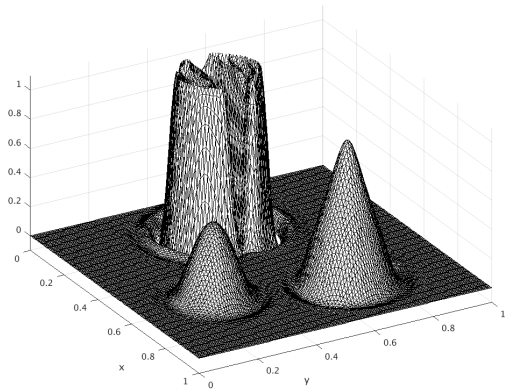
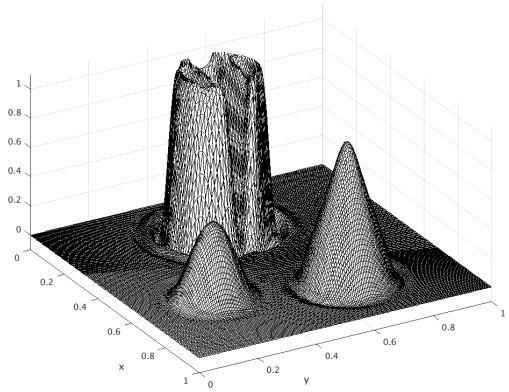
SUPG stabilization



Single Domain



MFR



No artifacts due to interface, but some oscillations expected with SUPG stabilization

Conclusions

- Confirmed the potential of Lagrange-multiplier formulation for partitioned solution of interface problems
- Key idea is to consider alternative constraint, which enables explicit treatment of Lagrange multiplier
- Framework provides a way to generate partitioned approaches that can be traced back to a well-posed system
- MFR and VFR show
  - second order convergence in numerical experiments
  - equivalence to a monolithic solution in the case of matching nodes
  - ability to pass patch test