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A Multiscale Control Volume Finite Element Method for the Drift-Diffusion Equations on Unstructured Grids

Pavel Bochev, Kara Peterson, M. Perego
Computational Mathematics
Xujiao Gao
Advanced Device Technologies

Sandia National Laboratories

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Outline

- **Survey of numerical methods for the drift-diffusion model**
  - Stabilized FE and CVFEM formulations (SUPG, CVFEM-SU)
  - The Finite Volume Scharfetter-Gummel formulation (FVM-SG)
  - The good the bad and the ugly
- **New approach: stabilization by H(curl)-lifting of edge currents**
  - Provides multidimensional extension of SG to unstructured quad grids
  - Directly approximates and upwinds the flux
- **Computational studies**
  - Standard advection test problems
  - PN Diode and N-channel MOSFET devices
- **Extensions**
  - Parameter-free stabilized Galerkin method
  - 2nd order accurate multi-scale flux approximation

The Drift-Diffusion Model

$$\begin{cases} \nabla \cdot (\lambda^2 \nabla \phi) - (p - n + C) = 0 \\ \partial_t n - \nabla \cdot \mathbf{J}_n + R(\phi, n, p) = 0 \\ \partial_t p + \nabla \cdot \mathbf{J}_p + R(\phi, n, p) = 0 \end{cases}$$

Electric potential

Electron continuity: *n*= electron concentration

Hole continuity: *p* = hole concentration

$$\begin{cases} \mathbf{E} = -\nabla \phi \\ \mathbf{J}_n = n\mu_n \mathbf{E} + D_n \nabla n \\ \mathbf{J}_p = p\mu_p \mathbf{E} - D_p \nabla p \end{cases}$$

Electric field

Electron current density

Hole current density

Drift

Diffusion

$$\mathbf{u}_n = \mu_n \mathbf{E}; \ \mathbf{u}_p = \mu_p \mathbf{E}$$

Carrier drift velocities

$$D_n << \mathbf{u}_n; \ D_n << \mathbf{u}_p$$

Strong drift regime

We use the electron equation to present the method development

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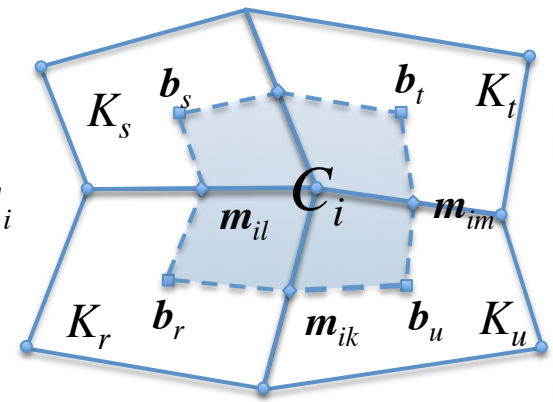
Stabilized Formulations

- **Stabilized Finite Element method (SUPG)**

$$\int_{\Omega}\left(\partial_t n + R(\phi,n,p)\right)\hat{n}\,dV + \int_{\Omega}\left(D_n \nabla n\cdot\nabla\hat{n} - (\mathbf{u}_n\cdot\nabla n)\hat{n}\right)dV + \int_{\Omega}\tau(\mathbf{u}_n\cdot\nabla n)(\mathbf{u}_n\cdot\nabla\hat{n})\,dV = 0 \quad \forall\hat{n}$$

- **Stabilized CVFEM (CVFEM-SU)**

$$\int_{C_i}\partial_t n + R(\phi,n,p)dV + \int_{\partial C_i}\mathbf{J}_n\cdot\mathbf{n}dS + \int_{\partial C_i}\tau\left(\mathbf{u}_n\nabla\cdot\left(\mathbf{u}_n n\right)\right)\cdot\mathbf{n}dS = 0 \quad \forall C_i$$



- **Use the same stabilization parameter**

$$\tau\big|_K = \frac{1}{\sqrt{\mathbf{u}_n^T\mathbf{G}_K\mathbf{u}_n}}\left(\coth Pe_K - \frac{1}{Pe_K}\right); \quad Pe_K = \frac{\sqrt{\mathbf{u}_n^T\mathbf{G}_K\mathbf{u}_n}}{D_n\left\|\mathbf{G}_K\right\|} \quad \forall K$$

- **Use the same C^0 nodal element for the carrier concentration**

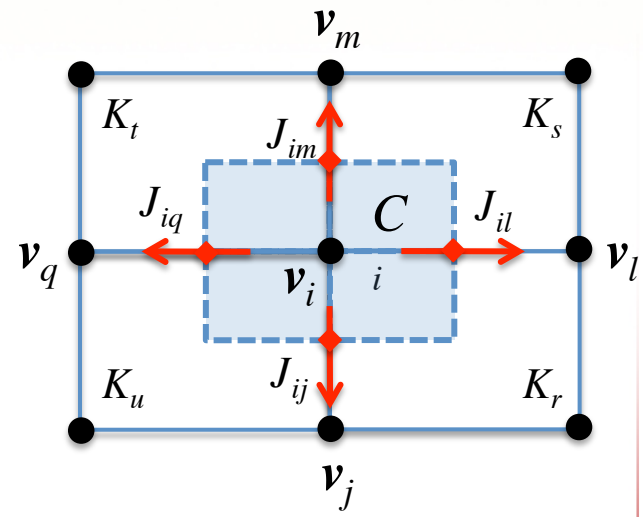
The Scharfetter-Gummel Finite Volume

Integrate continuity equation over ***topologically dual*** control volumes

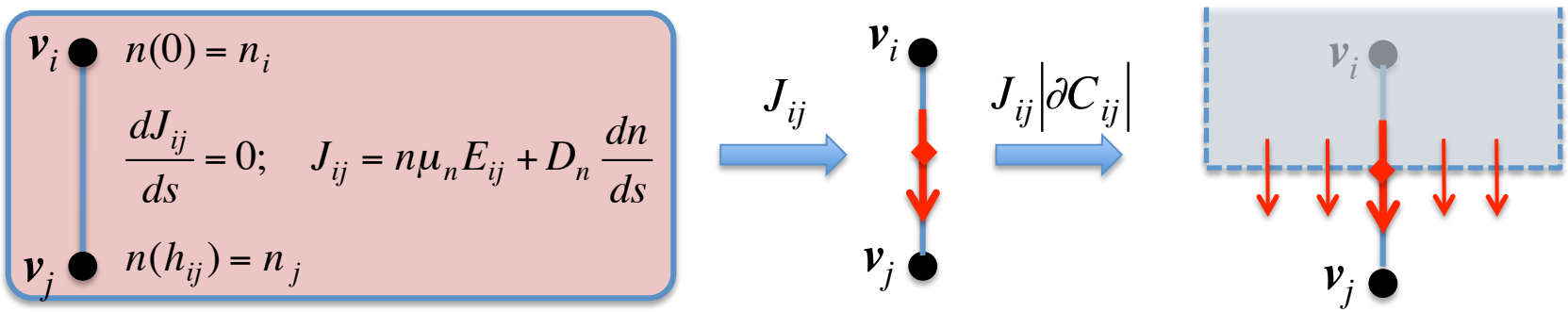
$$\int_{C_i} \partial_t n + R(\phi, n, p)\, dV + \int_{\partial C_i} \mathbf{J}_n \cdot \mathbf{n}\, dS = 0 \quad \forall C_i$$

$$n\big|_{C_i} \approx n_i \quad \forall C_i \qquad\qquad \mathbf{J}\cdot\mathbf{n}\big|_{m_{ij}} \approx J_{ij} \quad \forall \partial C_{ij}$$

$$\boxed{\left(\partial_t n_i + R(\phi_i, n_i, p_i)\right)|C_i| \;+\; \sum_{\partial C_{ij}} J_{ij}\left|\partial C_{ij}\right| = 0 \quad \forall C_i}$$



Scahrfetter-Gummel current estimates J_{ij}: assume J_{ij}=*const*



$$a_{ij} = \frac{h_{ij} E_{ij}}{2\beta} \qquad\qquad J_{ij} = \frac{a_{ij} D_n}{h_{ij}}\left[n_j\left(\coth(a_{ij}) + 1\right) - n_i\left(\coth(a_{ij}) - 1\right)\right]$$

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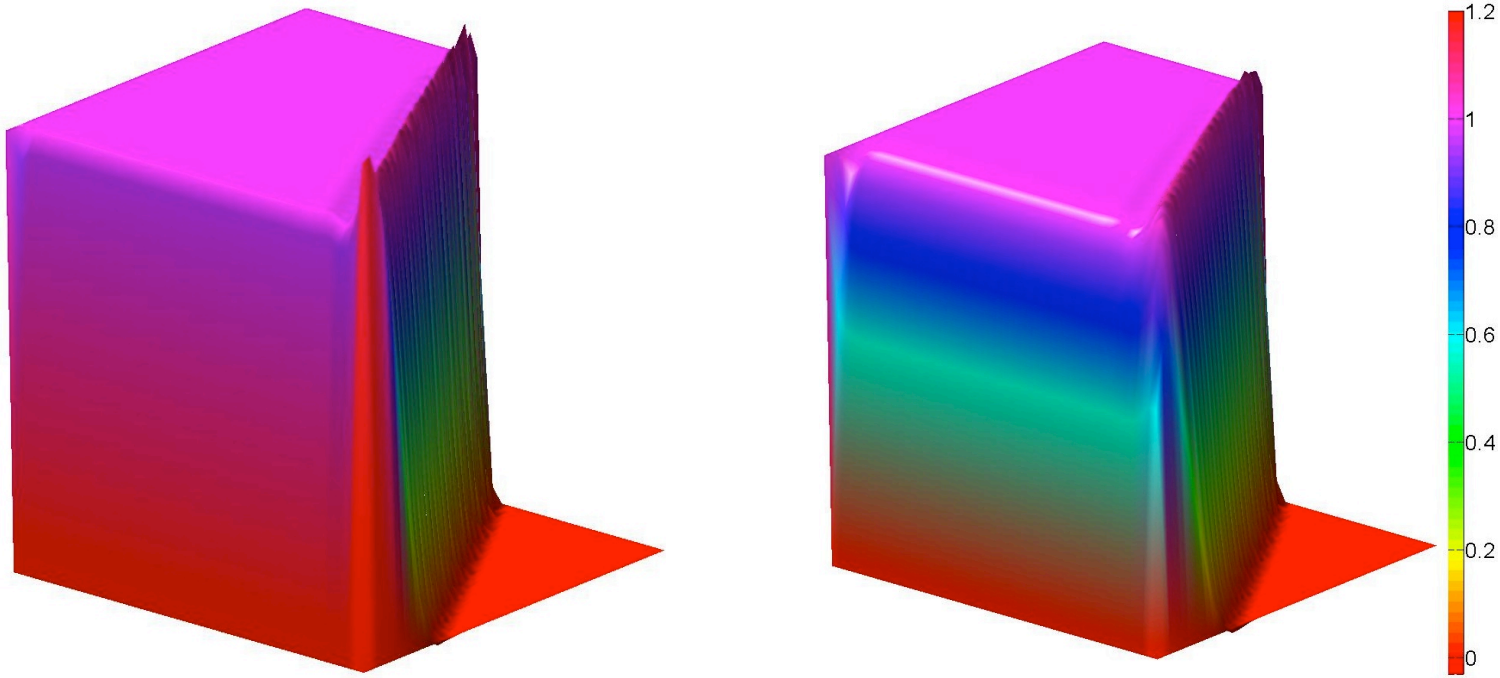
The good, the bad and the ugly

One Tau does not fit all: resolution of boundary vs. internal layers in SUPG & CVFEM-SU



$$\tau\big|_K = \frac{1}{\sqrt{\mathbf{u}_n^T \mathbf{G}_K \mathbf{u}_n}} \left(\coth Pe_K - \frac{1}{Pe_K} \right)$$

$$\tau_{\inf} = \frac{2h}{\sqrt{\mathbf{u}_n^T \mathbf{G}_K \mathbf{u}_n}}$$

P.Knobloch. On the definition of the SUPG parameter. ETNA, 32, 76–89,2008.

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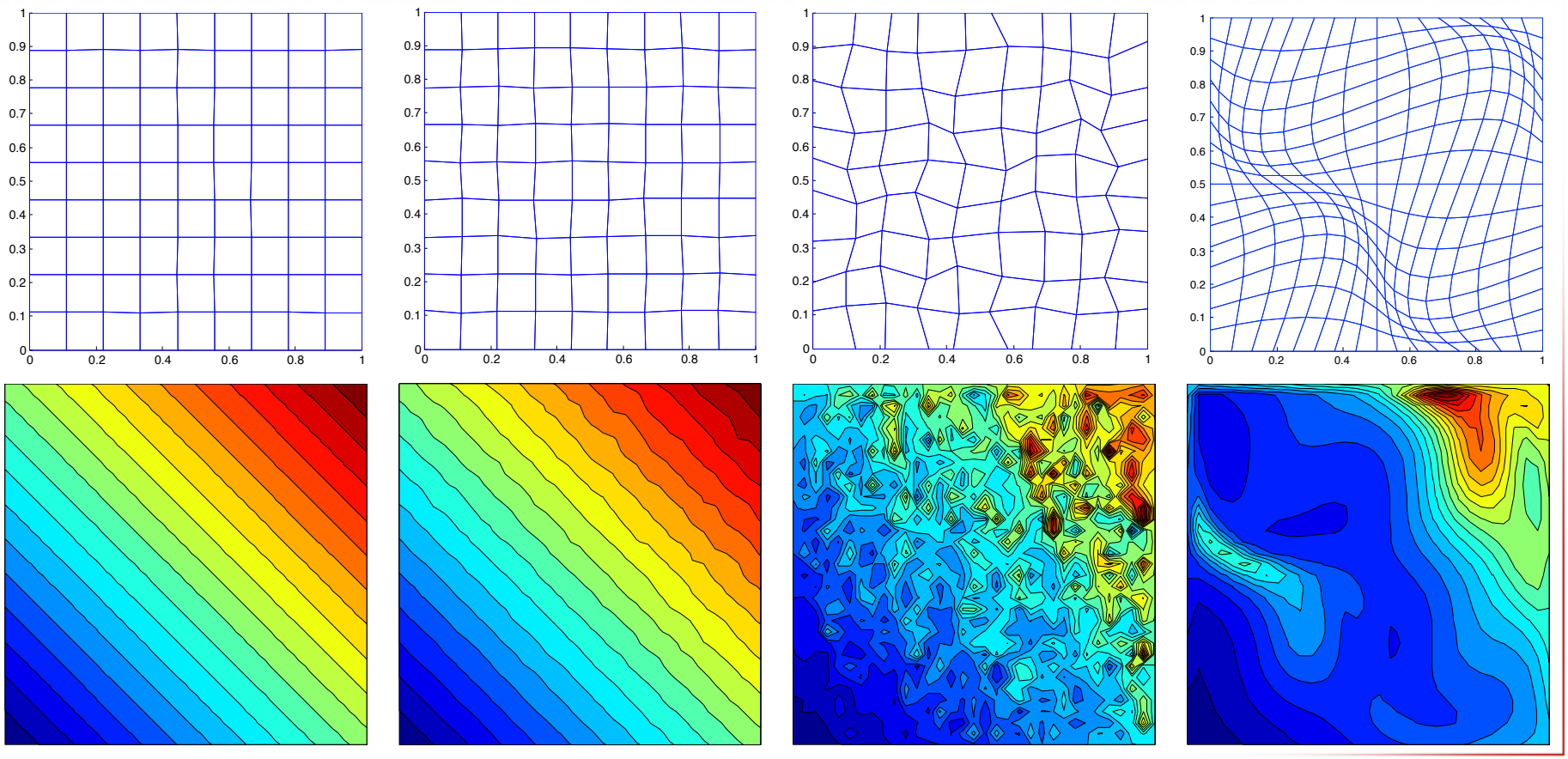
The good, the bad and the ugly

Approximation of a linear function by FVM-SG on grids with varying topological duality



\+ ← Topological duality → -

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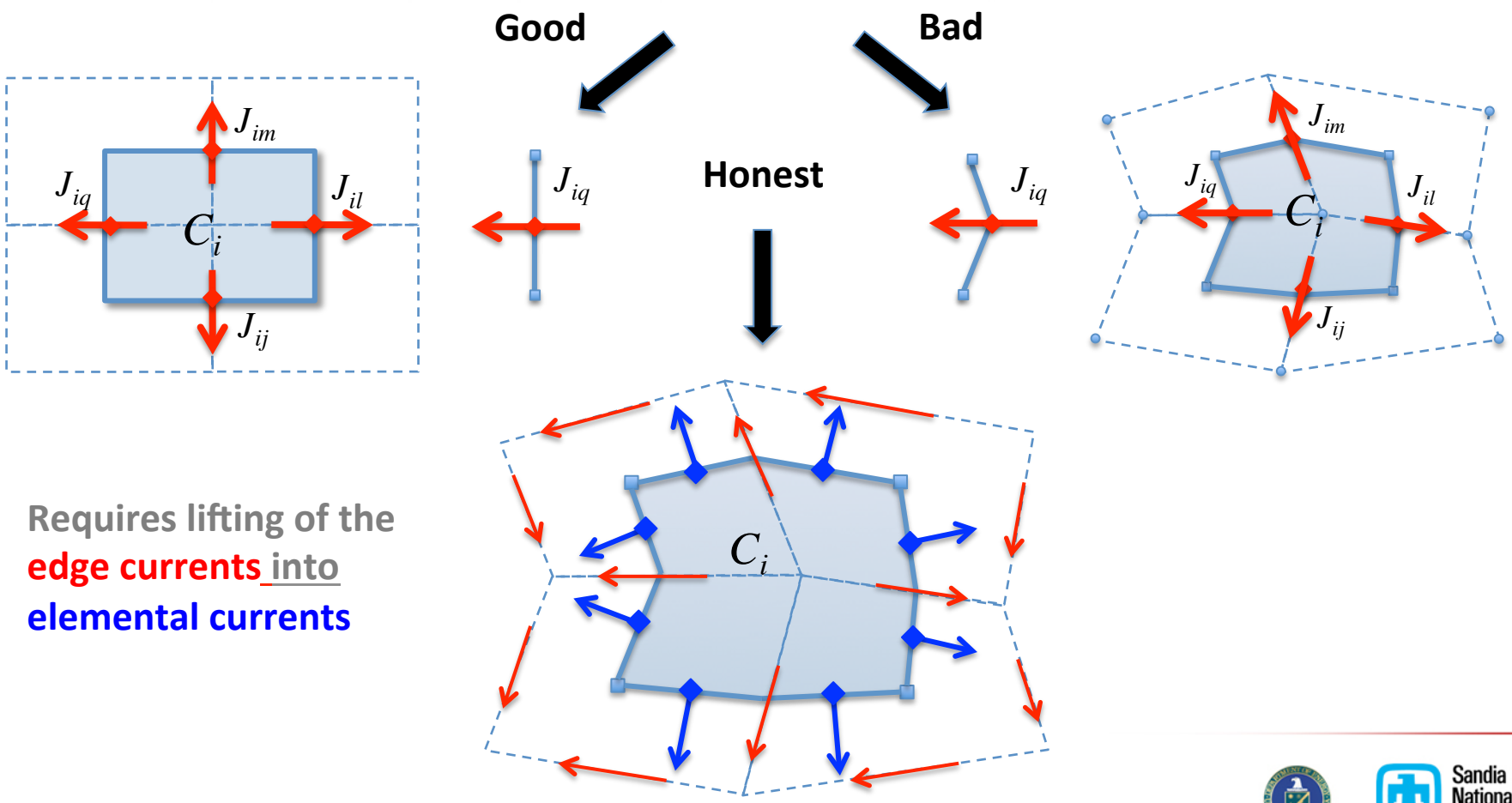
The good, the bad and the ugly

SUPG	CVFEM-SU	FVM-SG
Work on arbitrary unstructured grids		Requires topologically dual grids
Require tunable stabilization parameter		**No tunable parameters**
Violate physical solution bounds		**Essentially monotone**
Not conservative	**Conservative**	**Conservative**

- FVM-SG is the workhorse in virtually all commercial device simulators.
- However, FVM-SG fails on general unstructured grids
- **Goal: method that combines the best properties and works on arbitrary grids**

The challenge

Approximation of the flux integral: $\int_{\partial C_i} \mathbf{J}_n \cdot \mathbf{n} dS$



Requires lifting of the
edge currents into
elemental currents

The H(curl)-lifting approach

Use *H(curl)* conforming (edge) elements to extend edge currents into the elements

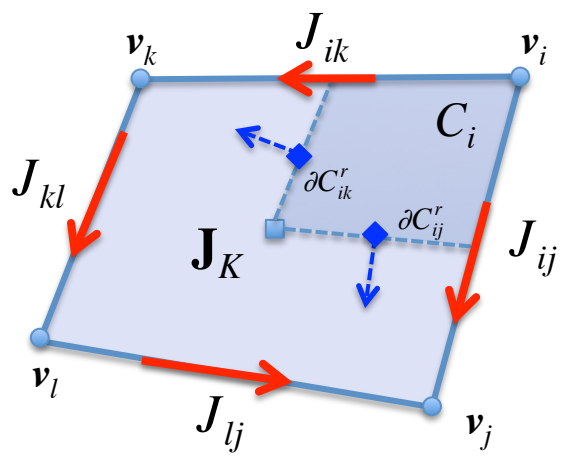
$$C^h(\Omega) = span\left\{\vec{W}_{ij}\right\}; \quad \int_{e_{ij}} \vec{W}_{ij} \cdot t_{rs} d\ell = \delta_{ij}^{rs}$$

Lowest-order Nedelec edge element basis

H(curl)–conforming lifting of the Scharfetter-Gummel edge currents

$$J_{ij} = \frac{a_{ij} D_n}{h_{ij}} \left[n_j \left(\coth(a_{ij}) + 1 \right) - n_i \left(\coth(a_{ij}) - 1 \right) \right]$$

$$\mathbf{J}_{SG} = \sum_{edges} h_{ij} J_{ij} \vec{W}_{ij}$$



The CVFEM-SG formulation

- Start with a standard nodal CVFEM formulation

$$\int_{C_i} \partial_t n + R(\phi, n, p)\, dV + \int_{\partial C_i} \mathbf{J}_n(n) \cdot \mathbf{n}\, dS = 0 \quad \forall C_i$$

- Swap the nodal flux with the *H(curl)* lifting of the SG edge currents

$$\mathbf{J}_n = n\mu_n \mathbf{E} + D_n \nabla n \quad \rightarrow \quad \mathbf{J}_{SG} = \sum_{edges} h_{ij} J_{ij} \vec{W}_{ij}$$

- Get the CVFEM-SG formulation

$$\int_{C_i} \partial_t n + R(\phi, n, p)\, dV + \int_{\partial C_i} \mathbf{J}_{SG} \cdot \mathbf{n}\, dS = 0 \quad \forall C_i$$

Why edge element lifting?

Let's examine the flux at the diffusive limit, i.e., when the drift velocity is zero

Theorem

Assume that the carrier drift velocity $\mathbf{u}_n = \mu_n \mathbf{E} = \mathbf{0}$. Then the nodal flux has the form

$$\mathbf{J}_n(n_h) = \sum_{edges} D_n (G_{ij}\mathbf{n}) \vec{W}_{ij} \quad \text{where} \quad G_{ij}\mathbf{n} = (n_j - n_i)$$

Row of the Node-to-Edge incidence matrix (**topological gradient**)

Proof: Let $\{N_i\}$ be the C^0 nodal basis, i.e., $n = \sum_{nodes} n_i N_i$.

$$\mu_n \mathbf{E} = \mathbf{0} \quad \Rightarrow \quad \mathbf{J}_n = D_n \nabla n = \sum_{nodes} D_n n_i \nabla N_i$$

No carrier drift

$$\nabla N_i = \sum_{edges} \sigma_{ij} \vec{W}_{ij} \quad \Rightarrow \quad \mathbf{J}_n = D_n \nabla n = \sum_{nodes} D_n n_i \sum_{edges} \sigma_{ij} \vec{W}_{ij}$$

Exact sequence property

nodes

edges

$$\Rightarrow \quad \mathbf{J}_n = \sum_{edges} D_n (n_i - n_j) \vec{W}_{ij} = \sum_{edges} D_n (G_{ij}\mathbf{n}) \vec{W}_{ij}$$

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Why edge element lifting?

Corollary

Assume that the carrier drift velocity $\mathbf{u}_n = \mu_n \mathbf{E} = \mathbf{0}$. Then the nodal flux has the form

$$\mathbf{J}_n(n_h) = \sum_{edges} h_{ij} J_{ij}^0 \vec{W}_{ij}$$

Where the edge currents J_{ij}^0 solve the edge ODEs

$$\frac{dJ_{ij}^0}{ds} = 0; \quad J_{ij}^0 = D_n \frac{dn}{ds}; \quad n(0) = n_i \text{ and } n(h_{ij}) = n_j$$

Proof. Solving the edge ODE yields

$$n(s) = n_i + s\frac{n_j - n_i}{h_{ij}} \quad \Rightarrow \quad J_{ij}^0 = D_n \frac{dn}{ds} = D_n \frac{n_j - n_i}{h_{ij}} \quad \Rightarrow \quad h_{ij} J_{ij}^0 = D_n (n_j - n_i)$$

Diffusive flux = the *H(curl)*-lifting of the Scharfetter-Gummel edge currents!

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Consistency of the H(curl) lifting

Lemma

Assume D_n and μ_n are constant and ψ is linear along the edges. In the pure diffusion limit

$$\lim_{\mathbf{u}_n \to 0} \mathbf{J}_{SG} = \mathbf{J}_n(n_h)$$

Proof.

$$\lim_{\mathbf{u}_n \to 0} J_{ij} = \lim_{a_{ij} \to 0} \frac{a_{ij} D_n}{h_{ij}} \left[n_j \left(\coth(a_{ij}) + 1 \right) - n_i \left(\coth(a_{ij}) - 1 \right) \right] = D_n \frac{n_j - n_i}{h_{ij}} = J_{ij}^0$$

CVFEM-SG as a Multiscale Method

The multiscale finite element basis $\{\phi_i\}$ (MFEM, Hou et al JCP 1997) is defined by solving element PDEs

$$\begin{cases} -\nabla \cdot \left(\phi_i \mu_n \mathbf{E} + D_n \nabla \phi_i \right) = 0 & \text{in } K_r \text{ and } \phi_i = \gamma_i \text{ on } K_r \\ -\dfrac{d}{ds} \left(\gamma_i \mu_n E_{kl} + D_n \dfrac{d\gamma_i}{ds} \right) = 0 & \text{on } e_{kl} \text{ and } \gamma_i(\mathbf{v}_k) = \delta_i^k; \quad \gamma_i(\mathbf{v}_l) = \delta_i^l \end{cases}$$

- ➠ MFEM: lifts the edge currents into **scalar basis functions by solving element PDEs**
- ➠ CVFEM-SG: lifts the edge currents into **an approximation of the current density directly**

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The stabilization mechanism

Theorem

Let I_E denote the edge element interpolant. Then

$$\mathbf{J}_{SG} = I_E(\mathbf{J}_n(n_h)) + \Theta(n_h); \qquad \Theta(n_h) = \sum_{edges} \theta_{ij}(G_{ij}\mathbf{n})\vec{W}_{ij}$$

Edge element interpolant of the (unstable) nodal flux

$$\mathbf{J}_{SG} = I_E(\mathbf{J}_n(n_h)) + \Theta(n_h)$$

Stabilizing artificial diffusion

Stabilizing edge diffusion

$$\Theta(n_h) = \sum_{edges} \theta_{ij}(G_{ij}\mathbf{n})\vec{W}_{ij}$$

$$\theta_{ij} = D_{ij}(a_{ij}\coth a_{ij} - 1)$$

Row of the Node-to-Edge incidence matrix (topological gradient)

Numerical results

- **Manufactured solution tests:**
  - Convergence rates
  - Robustness on non-affine grids
- **Steady state scalar advection tests**
  - Boundary + internal layer
- **Transient scalar advection tests**
  - Rotation of a cylinder
  - Skew advection of a cylinder
- **Device simulations (fully coupled nonlinear tests)**
  - PN diode
  - N-channel MOSFET device

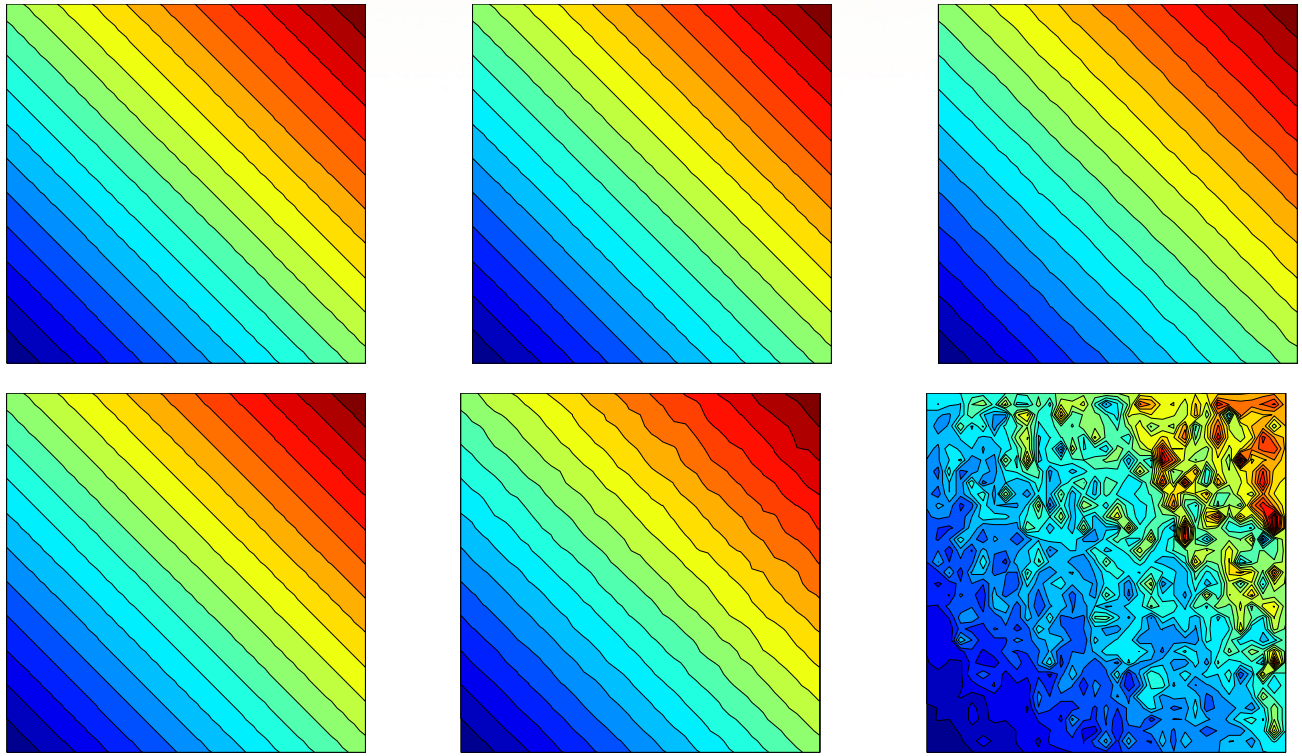
Convergence rates

	CVFEM-SG		FVM-SG		CVFEM-SU	
Error	*L2*	*H1*	*L2*	*H1*	*L2*	*H1*
Grid	***Moderate drift case***: D_n=1×10^{-3}					
32×32	0.4373E-02	0.7620E-01	0.4364E-02	0.7572E-01	0.5764E-02	0.1027E+00
64×64	0.2108E-02	0.4954E-01	0.2107E-02	0.4937E-01	0.2712E-02	0.6357E-01
128×128	0.9870E-03	0.3089E-01	0.9870E-03	0.3084E-01	0.1164E-02	0.3515E-01
Rate	**1.095**	**0.681**	**1.094**	**0.679**	**1.221**	**0.854**
Grid	***Strong drift case***: D_n=1×10^{-5}					
32×32	0.4732E-02	0.7897E-01	0.4723E-02	0.7850E-01	0.8161E-02	0.1310E+00
64×64	0.2517E-02	0.5477E-01	0.2515E-02	0.5460E-01	0.4194E-02	0.9312E-01
128×128	0.1298E-02	0.3834E-01	0.1298E-02	0.3828E-01	0.2122E-02	0.6590E-01
Rate	**0.955**	**0.514**	**0.955**	**0.514**	**0.983**	**0.499**

Robustness: random grids

CVFEM-SG

FVM-SG

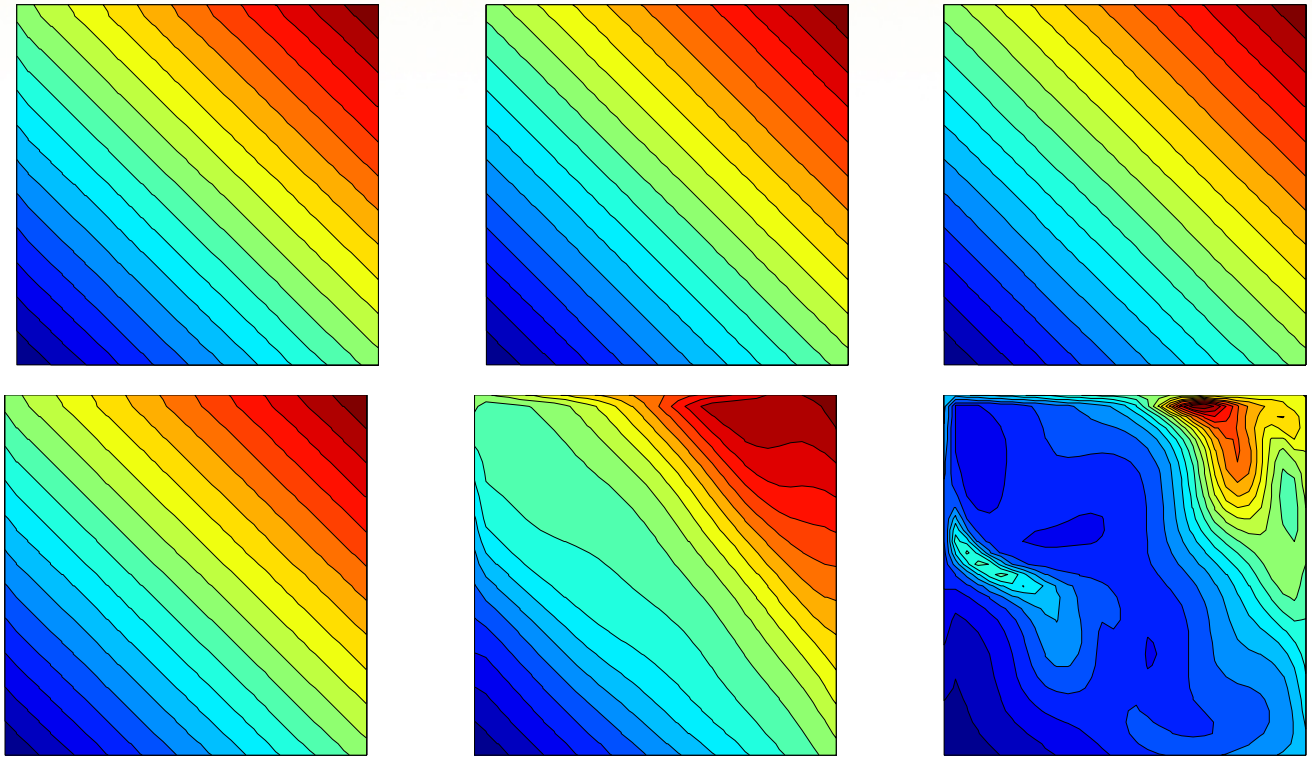


(a) $O(h^2)$ (b) $O(h)$ (c) $O(1)$

Robustness: smooth non-affine grid

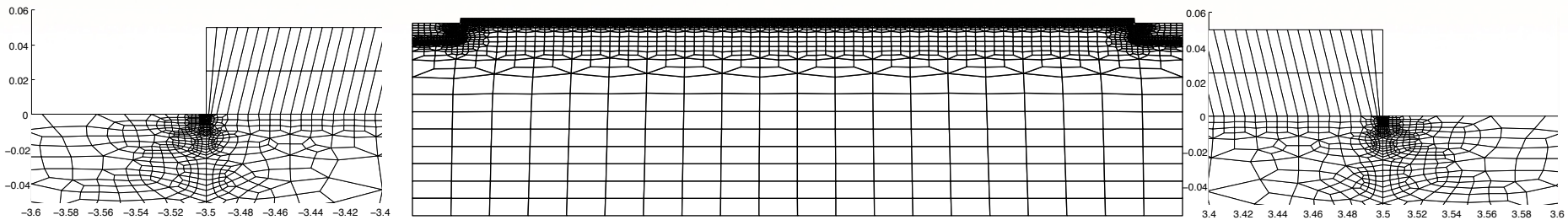
CVFEM-SG

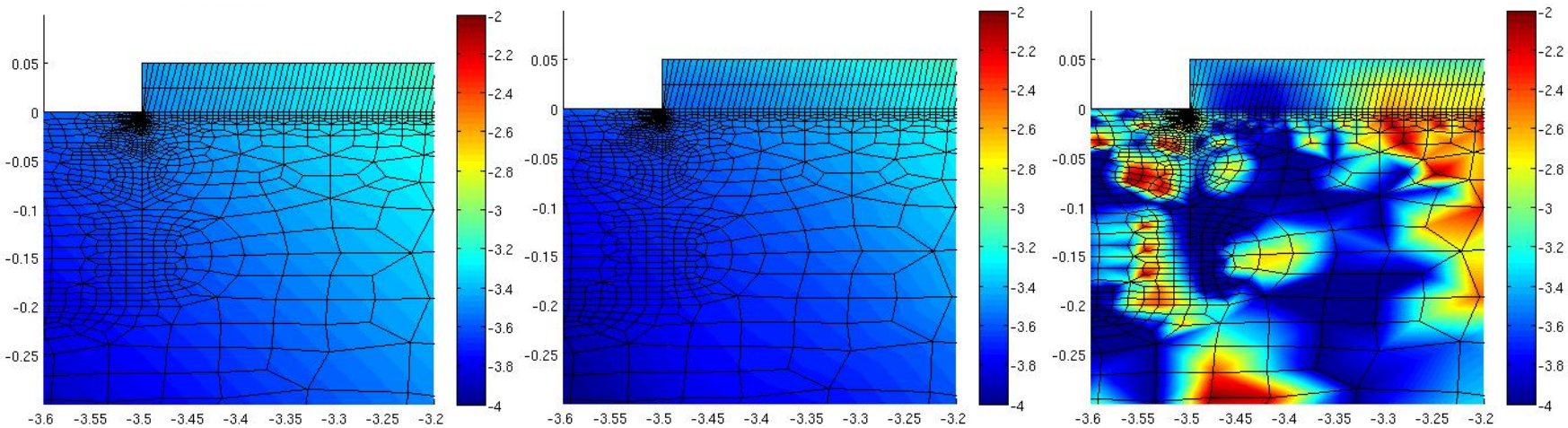
FVM-SG



(a) tensor grid (b) smooth grid $\gamma = 0.1$ (c) smooth grid $\gamma = 0.5$

Robustness: MOSFET grid

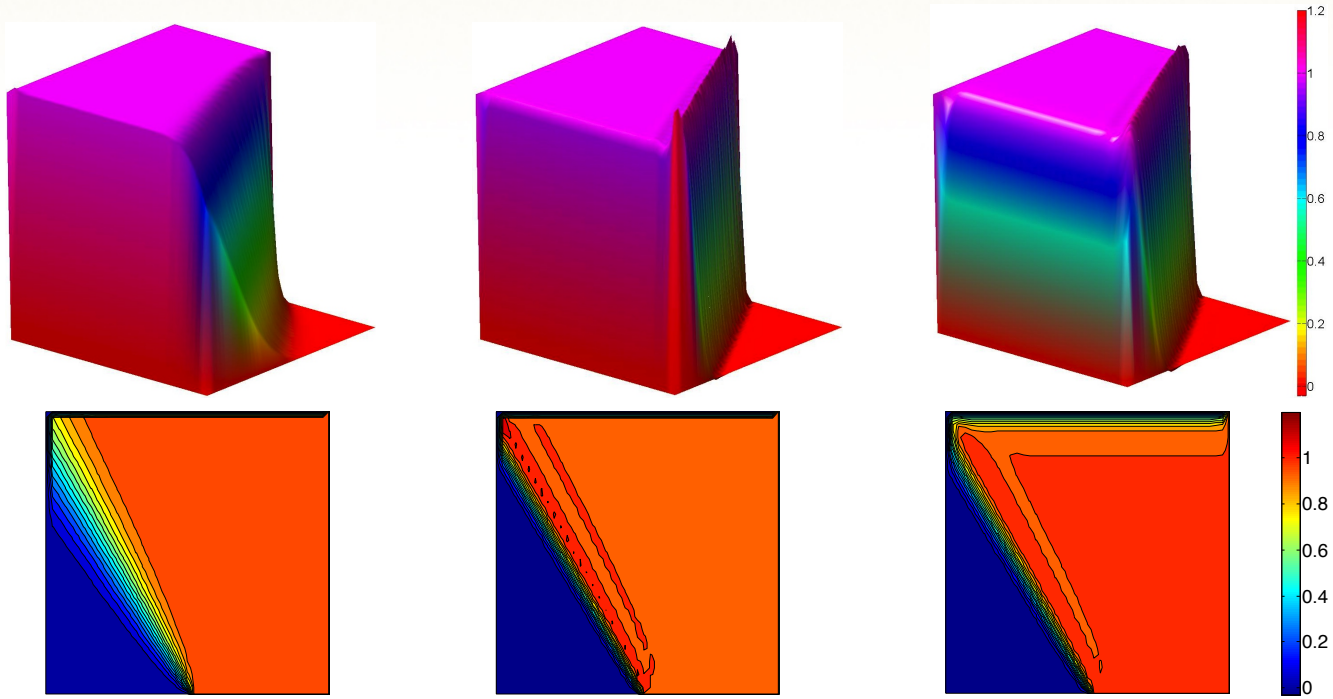




(a) Exact (b) CVFEM-SG (c) FVM-SG

Steady-state scalar advection

64×64 O(h) random grid



(a) CVFEM-SG (b) CVFEM-SU with (40) (c) CVFEM-SU with τ_{inf}

DMP

		Exact		CVFEM-SG		CVFEM-SU		CVFEM-SU$_{\text{inf}}$	
Grid	Pe	min	max	min	max	min	max	min	max
unif	781	0.00	1.00	0.00	1.06	-0.03	**1.30**	-0.04	1.08
O(h)	787	0.00	1.00	0.00	1.02	-0.03	**1.16**	-0.03	1.07

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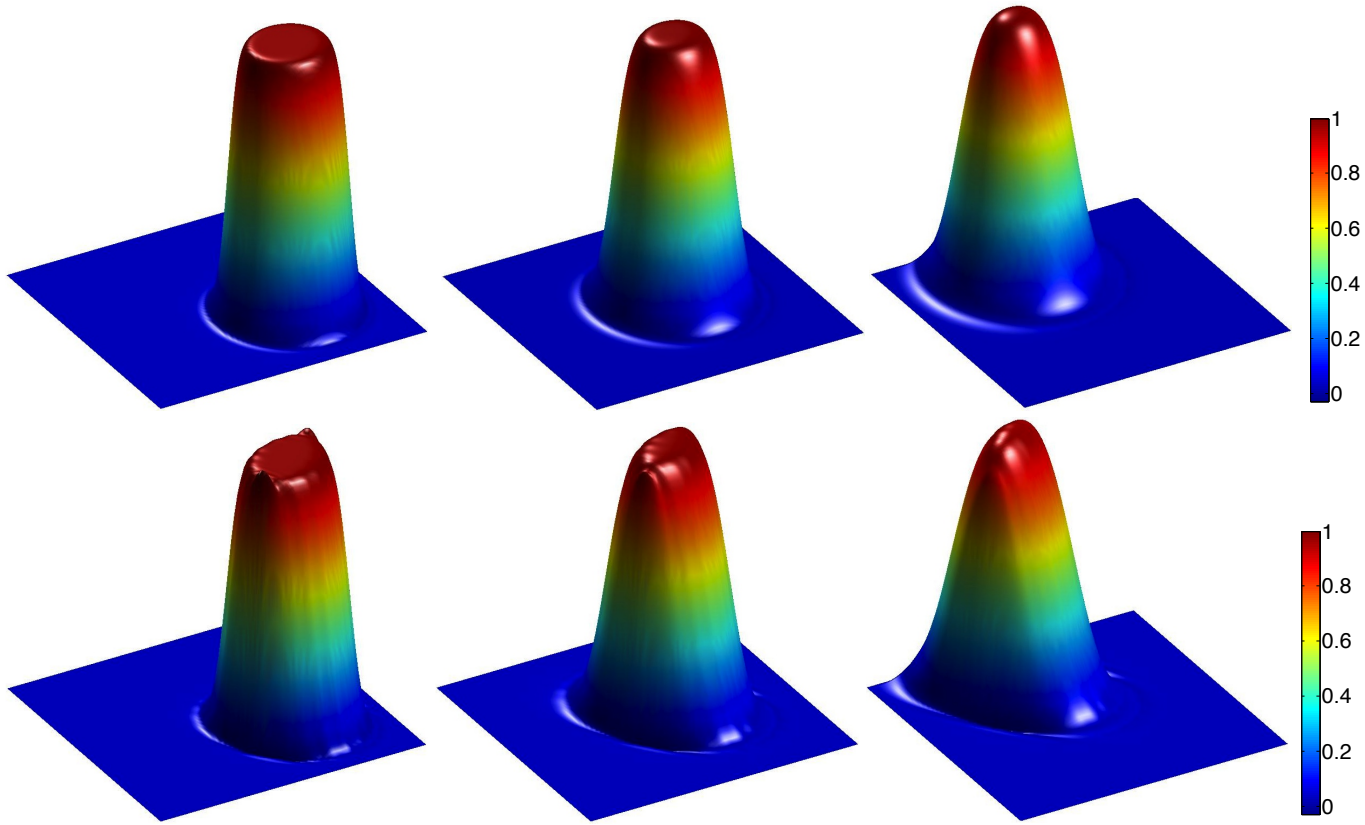
Transient scalar problem: skew advection

CVFEM-SG

CVFEM-SU

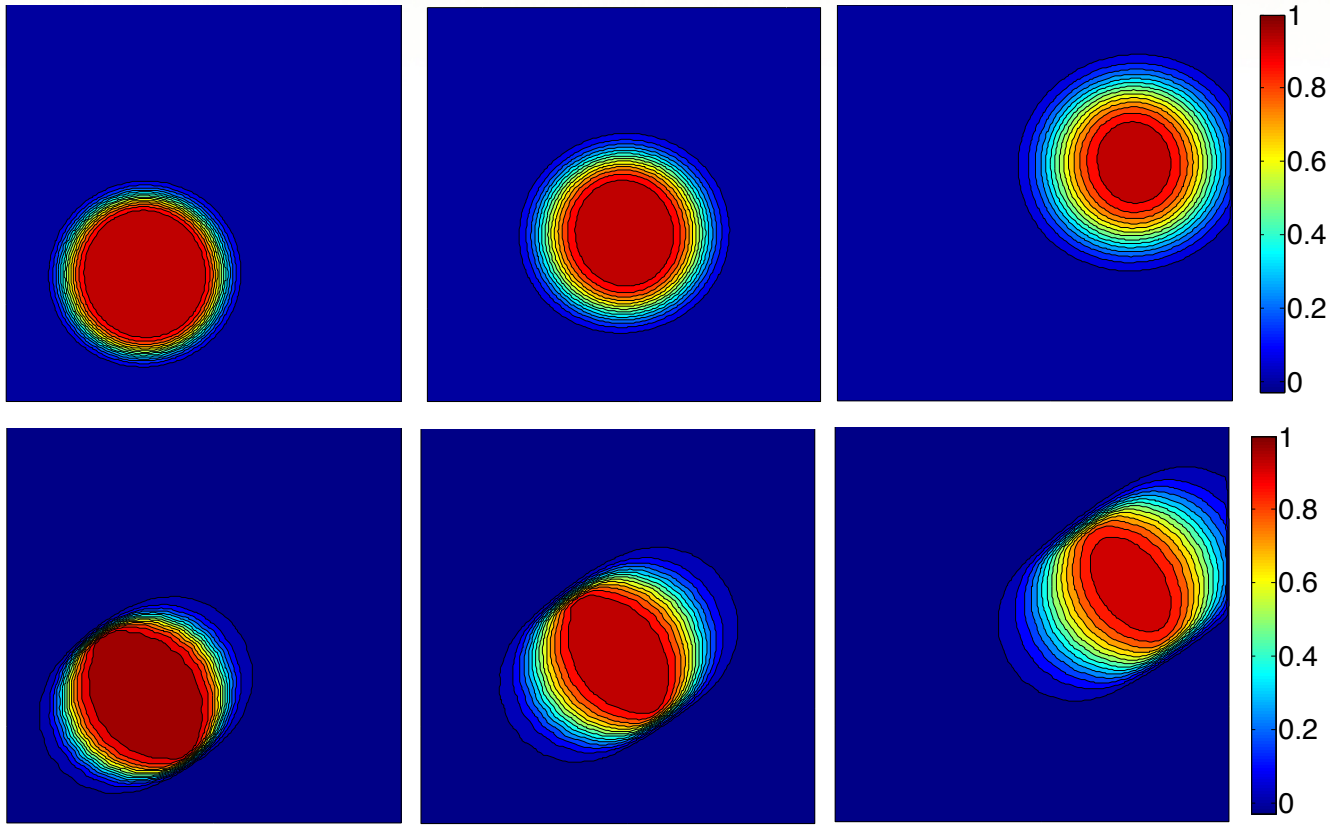


(a) $t = 0.1$ (b) $t = 0.25$ (c) $t = 0.5$

Transient scalar problem: skew advection

CVFEM-SG

CVFEM-SU

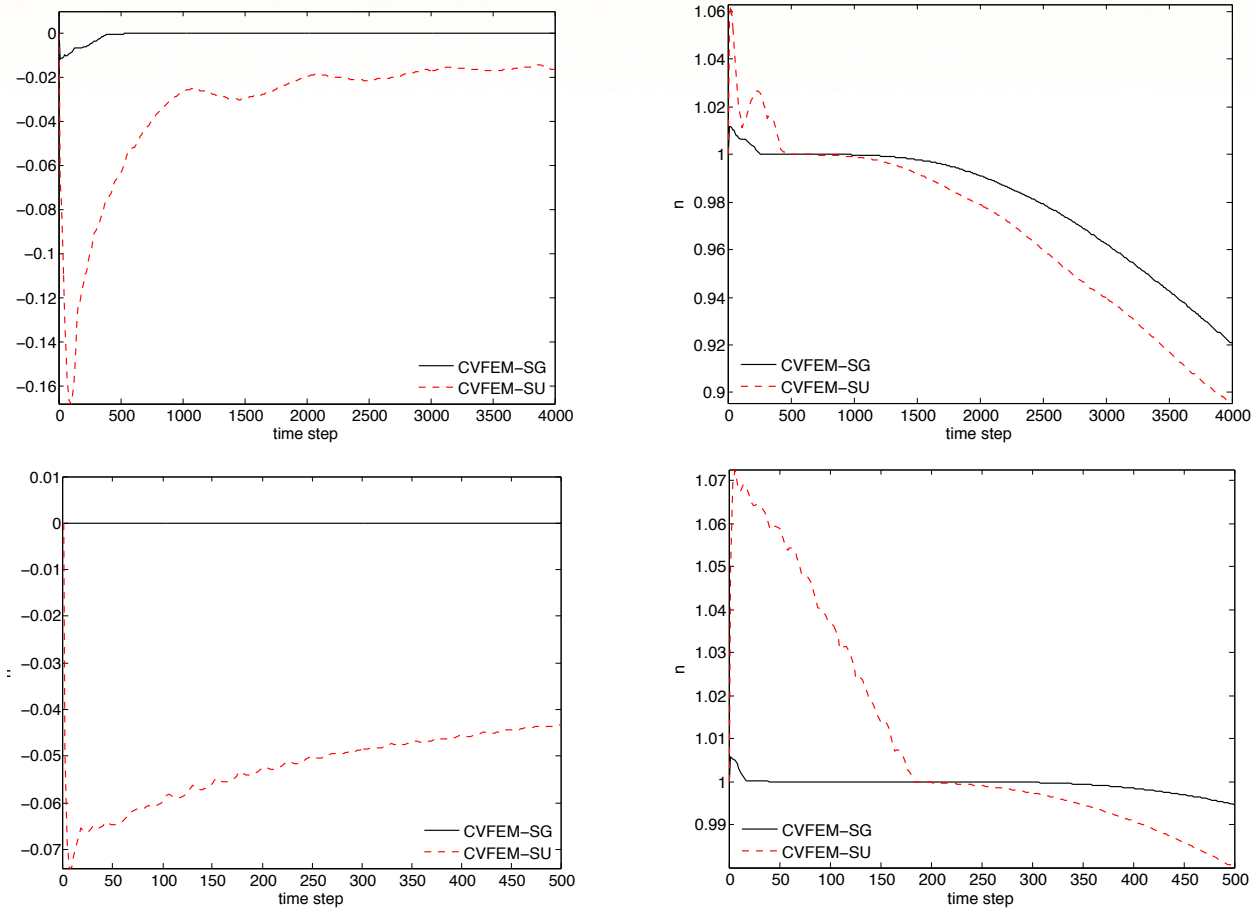


(a) $t = 0.1$ (b) $t = 0.25$ (c) $t = 0.5$

Transient problems: DMP

Rotation

Skew advection

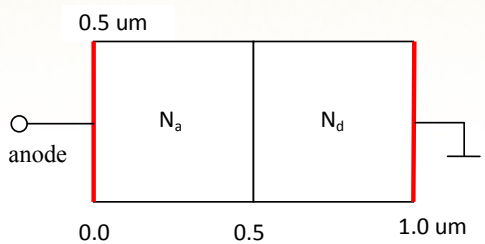


(a) Global lower bound

(b) Global upper bound

PN diode simulations

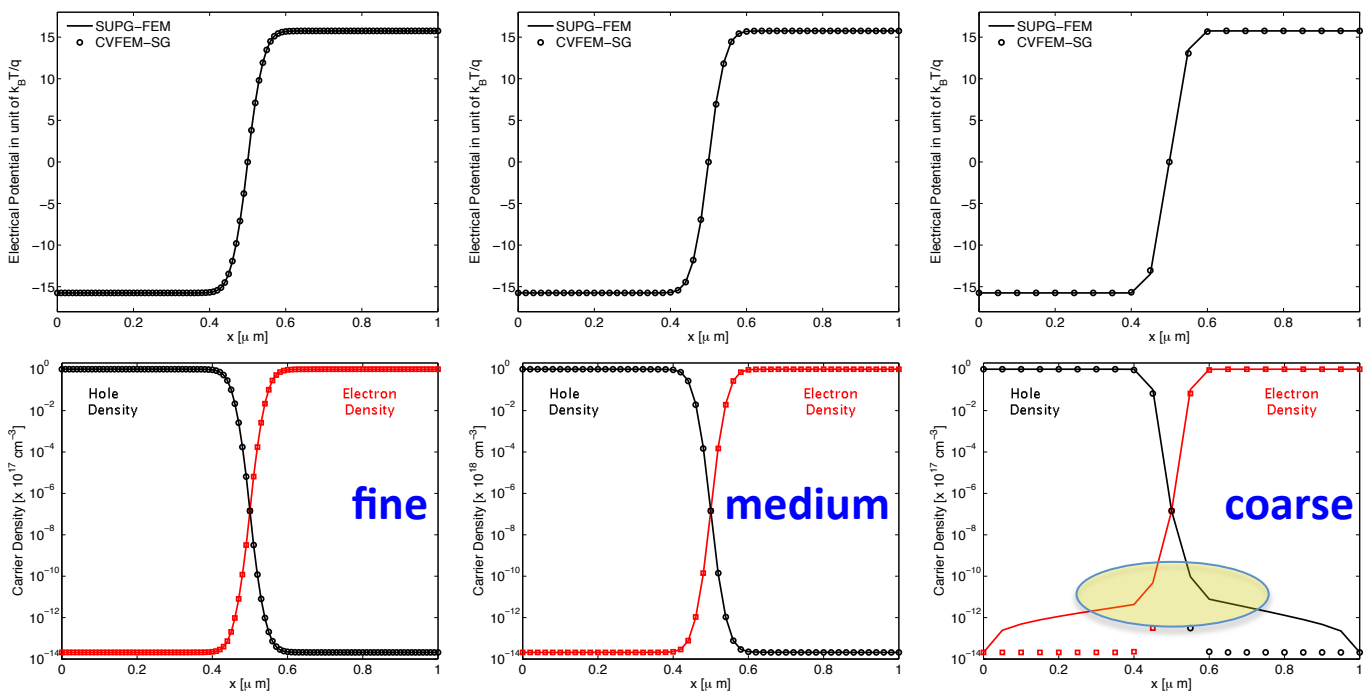
Model



$$R = \frac{np - n_i^2}{\tau_p(n + n_i) + \tau_n(p + n_i)} + (c_n n + c_p p)(np - n_i^2)\,.$$

Shockley-Read-Hall **Auger**

Mesh dependence study: CVFEM-SG (dots) vs. SUPG (solid lines)



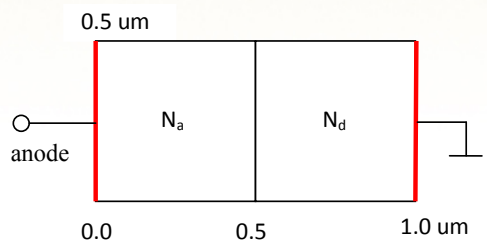
(a) $h_x = 0.01\mu m$ (b) $h_x = 0.02\mu m$ (c) $h_x = 0.05\mu m$

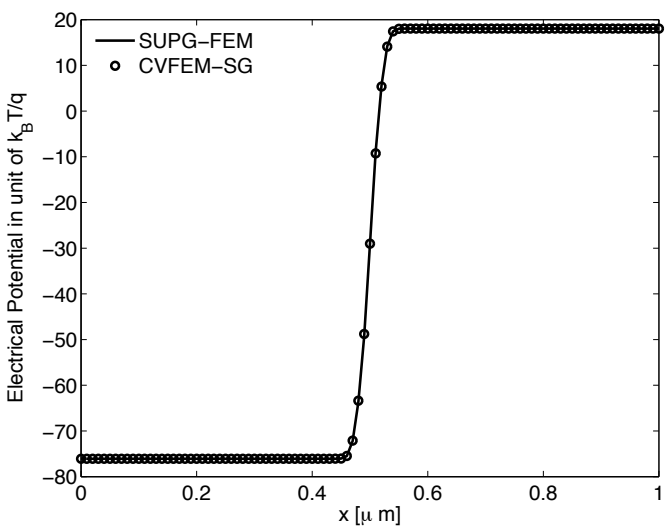
Potential

Carrier density

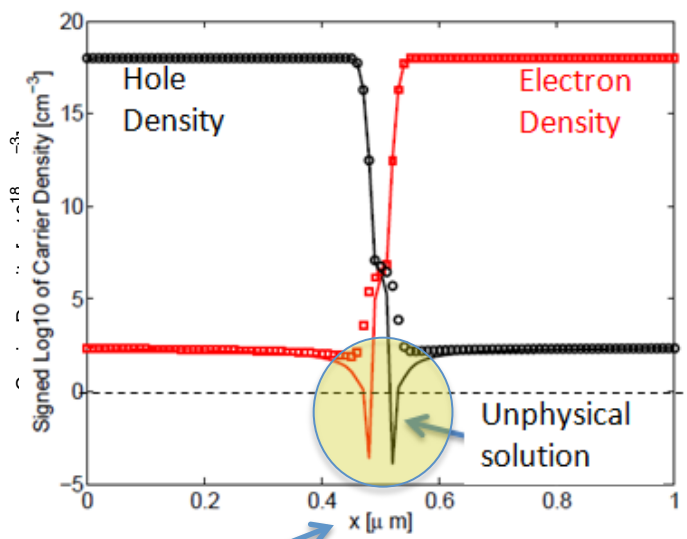
PN diode simulations

Strong drift study **-1.5V**





(a) ψ



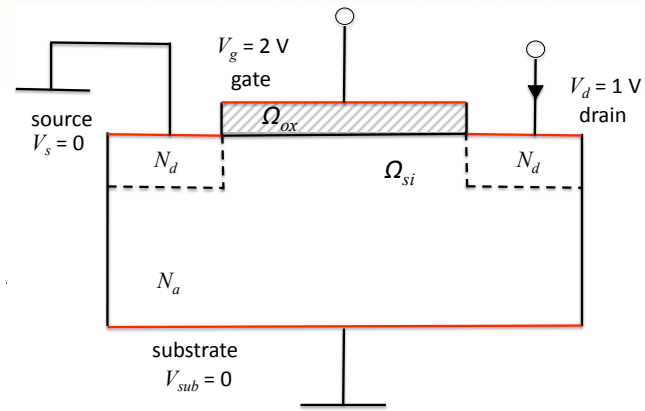
(b) n and p

Violation of physical bounds by FEM-SUPG

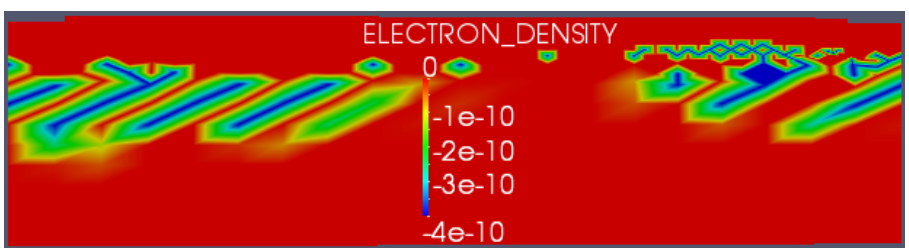
N-channel MOSFET simulations

Model

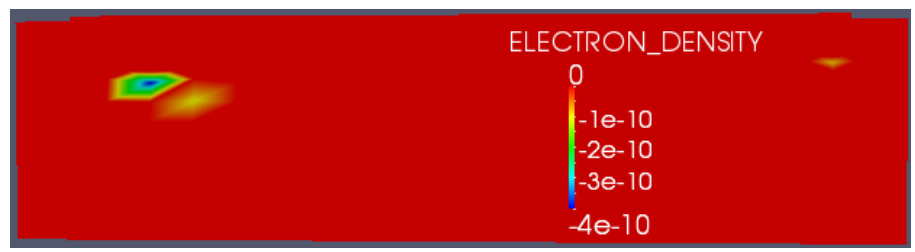
$$\nabla \cdot (\epsilon_0 \epsilon_{ox} \nabla \psi) = 0 \quad \text{in} \quad \Omega_{ox}$$
$$\nabla \cdot (\epsilon_0 \epsilon_{si} \nabla \psi) = -q\left(n_i \exp\left(\frac{-q\psi}{k_B T}\right) - n + N_d - N_a\right) \quad \text{in} \quad \Omega_{si}$$
$$\nabla \cdot (n\mu_n \nabla \psi - D_n \nabla n) = 0 \quad \text{in} \quad \Omega_{si}\,.$$



Violation of the lower physical bound



FEM-SUPG



CVFEM-SG

Extensions: Parameter-free stabilized Galerkin

Recall the stabilization mechanism of the elemental H(curl) flux:

$$\mathbf{J}_{SG} = I_E(\mathbf{J}_n(n_h)) + \Theta(n_h)$$

Edge element interpolant
of the (unstable) nodal flux

$$\begin{cases} \Theta(n_h) = \sum_{edges} \theta_{ij}(G_{ij}\mathbf{n})\vec{W}_{ij} \\ \theta_{ij} = D_{ij}(a_{ij}\coth a_{ij} - 1) \end{cases}$$

Artificial diffusion

Edge diffusion

How can we use this in a Galerkin method?

1. Discard the interpolant of the unstable nodal flux
2. Use artificial diffusion term to define a symmetric diffusion form
3. Augment standard Galerkin formulation by this form

Parameter-free stabilized Galerkin

Start with a standard Galerkin formulation

$$a(n_h, m_h) := \int_\Omega F(n_h) \cdot \nabla m_h \, dx = \int_\Omega f m_h \, dx$$

Combine with a symmetric artificial diffusion form

$$Q(n_h, m_h) := \int_\Omega \tilde{\Theta}(n_h) \cdot \tilde{\Theta}(m_h) \, dx \longrightarrow \begin{cases} \tilde{\Theta}(n_h) = \sum_{edges} \tilde{\theta}_{ij} (G_{ij}\mathbf{n}) \vec{W}_{ij} \\ \tilde{\theta}_{ij} = \sqrt{D_{ij}(a_{ij} \coth a_{ij} - 1)} \end{cases}$$

Get a new stabilized method

$$\underbrace{a(n_h, m_h) + Q(n_h, m_h)}_{a^h(n_h, m_h)} = \int_\Omega f m_h \, dx$$

Analysis

Theorem (Stability)

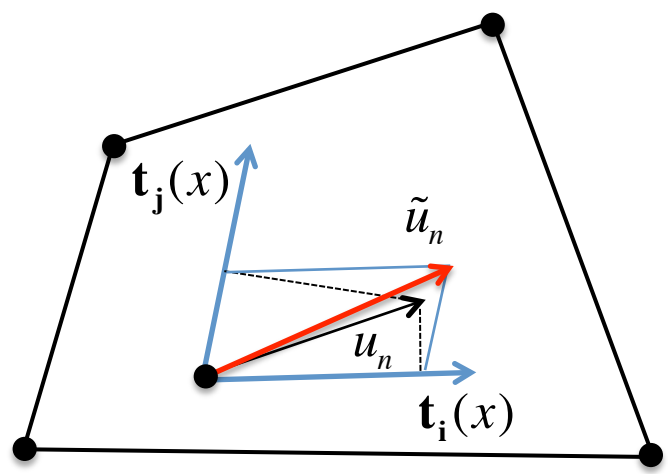
There exists ρ independent of ε, h and $\boldsymbol{u}_n$ such that

$$a^h(n_h, n_h) \geq \rho h \sum_{elements} |u_n| \, \|\nabla n_h\|_{0,k}^2$$ on triangles and tets

$$a^h(n_h, n_h) \geq \rho h \sum_{elements} \|\tilde{u}_n \cdot \nabla n_h\|_{0,k}^2$$ on quads and hexes

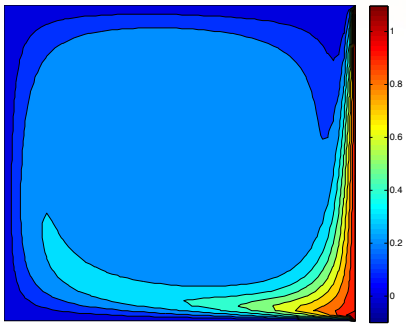
Effective velocity field on quad/hex element k

$$\tilde{u}_n(x) = \begin{cases} u_n \cdot \mathbf{t_i}(x) + u_n \cdot \mathbf{t_j}(x) \\ u_n \cdot \mathbf{t_i}(x) + u_n \cdot \mathbf{t_j}(x) + u_n \cdot \mathbf{t_k}(x) \end{cases}$$



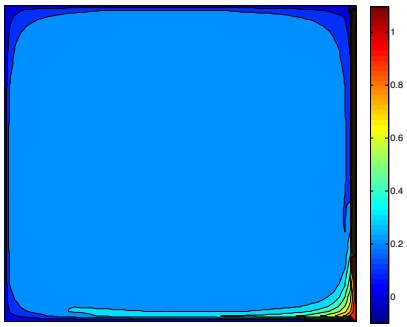
Analysis

Stability estimate implies the method is more diffusive on triangles and tets



Triangle grid

Quad grid



Theorem (Error estimates)

$$\sqrt{\varepsilon}\left\|n-n_h\right\|_1 \le C\sqrt{h}\left\|n\right\|_2 \qquad \left\|n-n_h\right\|_0 \le Ch\sqrt{\frac{h}{\varepsilon}}\left\|n\right\|_2$$

Actual L2 errors less sensitive to small diffusivity than suggested by this estimate

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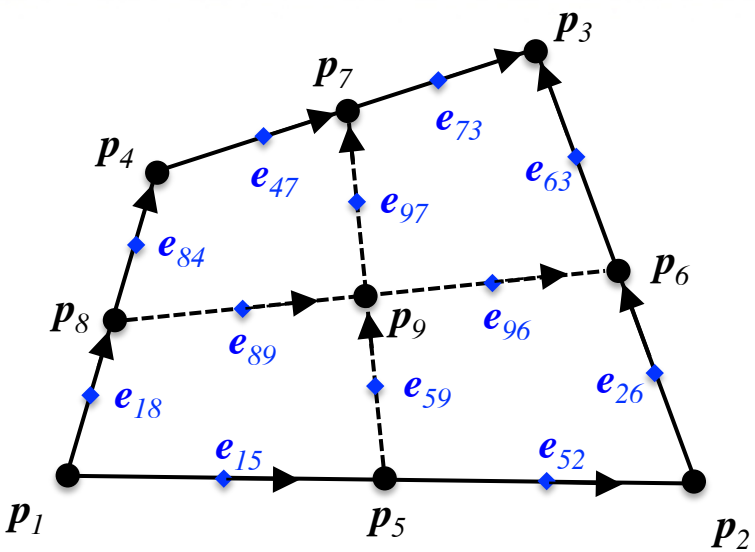
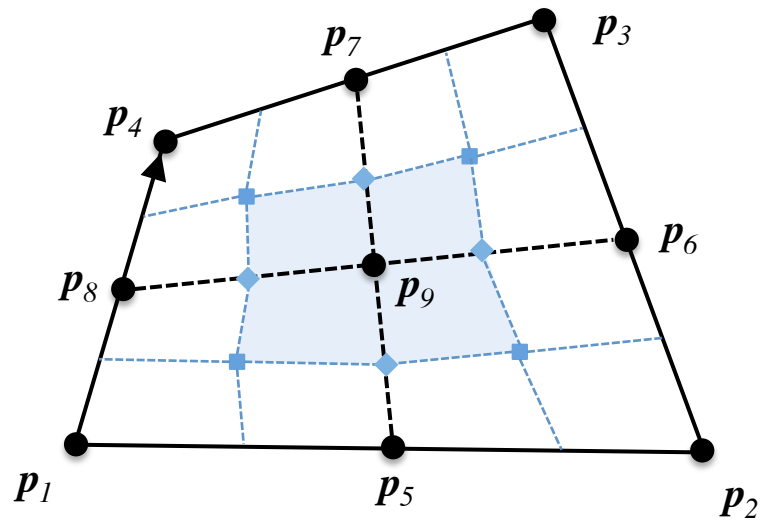
Extension to a 2nd order accurate formulation

Requires 2nd order Nedelec elements: 12 DoF associated with element sub-edges



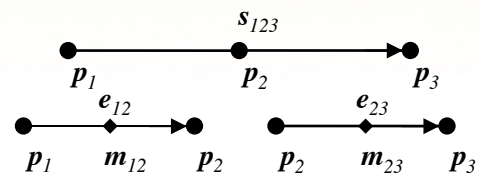


- Divide each element into 4 sub-elements
- Define control volumes for each sub-element node
- Define 2nd order fluxes F_{ij} on element sub-edges
- Use 2nd order edge elements to lift F_{ij} into an elemental flux F_h

Multi-scale sub-edge fluxes

Model advection-diffusion equation: $\longrightarrow$ $\nabla \cdot F(\phi) = f$ where $F(\phi) = \varepsilon \nabla \phi - \mathbf{u}\phi$

A mesh segment comprising two sub-edges $\longrightarrow$



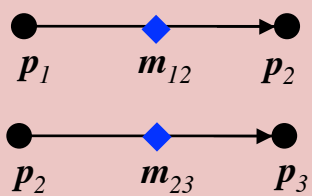
A segment flux of a real function $\phi : s_\alpha \rightarrow \mathbf{R}$ $\longrightarrow$ $F_\alpha(s) = \varepsilon_\alpha \phi' - u_\alpha \phi$

Assume $\phi(s)$ is such that

$\phi(0) = \phi_1 \quad \phi(h/2) = \phi_2 \quad \phi(h) = \phi_3$ $\longrightarrow$ Interpolates nodal values along S_{123}

$F_\alpha(s) = A + Bs$ $\longrightarrow$ Its segment flux is a linear function

Define sub-edge fluxes according to



$F_{12} = F_\alpha(m_{12})$ $\longrightarrow$ $\varepsilon_\alpha \phi'(h/4) - u_\alpha \phi(h/4)$

$F_{23} = F_\alpha(m_{23})$ $\longrightarrow$ $\varepsilon_\alpha \phi'(3h/4) - u_\alpha \phi(3h/4)$

Connection with Scharfetter-Gummel fluxes

Theorem

$$F_{12} = \Phi(\phi_1,\phi_2) + \gamma(\vec{\phi}) \qquad \Phi(\phi_1,\phi_2) = \frac{u_\alpha}{2}\left(\phi_2(\coth q_\alpha - 1) - \phi_1(\coth q_\alpha + 1)\right)$$

where

$$F_{23} = \Phi(\phi_2,\phi_3) + \gamma(\vec{\phi}) \qquad \Phi(\phi_2,\phi_3) = \frac{u_\alpha}{2}\left(\phi_3(\coth q_\alpha - 1) - \phi_2(\coth q_\alpha + 1)\right)$$

and

$$\gamma(\vec{\phi}) = \frac{\varepsilon_\alpha}{h}(\coth q_\alpha - 1)\left(\phi_1(\coth q_\alpha + 1) - 2\phi_2 \coth q_\alpha + \phi_3(\coth q_\alpha - 1)\right)$$

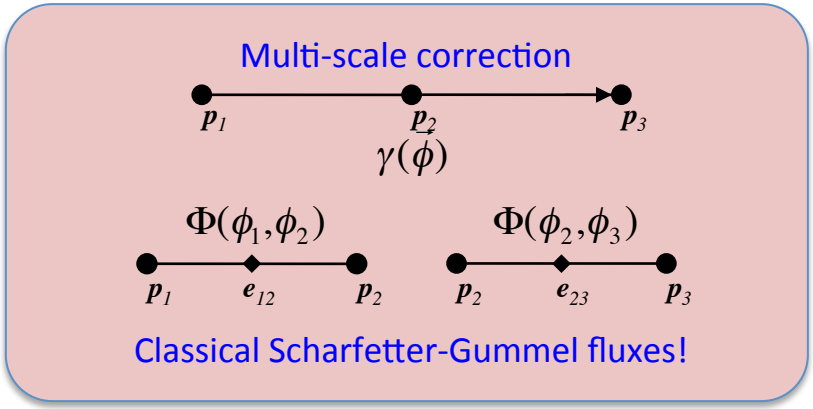
Proof: $\phi(s) = C_1(\vec{\phi})\exp(su_\alpha/\varepsilon_\alpha) + C_2(\vec{\phi}) + sC_3(\vec{\phi}) \quad \Rightarrow F(s) = -u_\alpha C_2(\vec{\phi}) + (\varepsilon_\alpha - su_\alpha)C_3(\vec{\phi})$

$$C_2(\vec{\phi}) = \phi_1 - \frac{\phi_1 - 2\phi_2 + \phi_3}{\left(\exp p_\alpha - 1\right)^2}$$

where

$$C_3(\vec{\phi}) = -2\frac{\left(\exp p_\alpha\right)\phi_1 - \left(\exp p_\alpha - 1\right)\phi_2 + \phi_3}{\left(\exp p_\alpha - 1\right)^2}$$

$$p_\alpha = \frac{u_\alpha h}{2\varepsilon_\alpha}; \quad q_\alpha = p_\alpha/2$$



Stabilization sources

Scharfetter-Gummel fluxes comprise flux approximation + diffusive flux term

$$\Phi(\phi_1,\phi_2) = f_{12} + \boxed{\varepsilon_\alpha q_\alpha (\coth q_\alpha - 1)\frac{\phi_2 - \phi_1}{h/2}}$$

$$f_{12} = -u_\alpha \frac{\phi_1 + \phi_2}{2} + \varepsilon_\alpha \frac{\phi_2 - \phi_1}{h/2} \approx F(m_{12})$$

$$\Phi(\phi_2,\phi_3) = f_{23} + \boxed{\varepsilon_\alpha q_\alpha (\coth q_\alpha - 1)\frac{\phi_3 - \phi_2}{h/2}}$$

$$f_{12} = -u_\alpha \frac{\phi_2 + \phi_3}{2} + \varepsilon_\alpha \frac{\phi_3 - \phi_2}{h/2} \approx F(m_{23})$$

Multi-scale correction term comprises diffusive + anti-diffusive flux terms

$$\gamma(\vec{\phi}) = \boxed{-\varepsilon_\alpha (q_\alpha \coth q_\alpha - 1)\frac{\phi_3 - \phi_1}{h}} + \varepsilon_\alpha h \coth q_\alpha (\coth q_\alpha - 1)\frac{\phi_1 - 2\phi_2 + \phi_3}{h^2}$$

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Convergence rates

	CVFEM-SG		CVFEM-SU		SUPG	
Error	*L2*	*H1*	*L2*	*H1*	*L2*	*H1*
Subelements	***Moderate drift case**: D_n=1×10^{-3}*					
32×32	0.1128E-01	0.6921E+00	0.7628E-01	0.9576E+00	0.2974E-02	0.3894E+00
64×64	0.2346E-02	0.3188E+00	0.3428E-01	0.4843E+00	0.7080E-03	0.1941E+00
128×128	0.4395E-03	0.1331E+00	0.1352E-01	0.2264E+00	0.2014E-03	0.9703E-01
Rate	**2.28**	**1.16**	**1.27**	**1.19**	**1.43**	**1.00**
Grid	***Strong drift case**: D_n=1×10^{-5}*					
32×32	0.1357E-01	0.7600E+00	0.8535E-01	0.1075E+01	0.3047E-02	0.3912E+00
64×64	0.3287E-02	0.3842E+00	0.4273E-01	0.6043E+00	0.6727E-03	0.1946E+00
128×128	0.8054E-03	0.1927E+00	0.2130E-01	0.3569E+00	0.1589E-03	0.9714E-01
Rate	**2.06**	**1.00**	**1.00**	**0.76**	**2.08**	**1.00**

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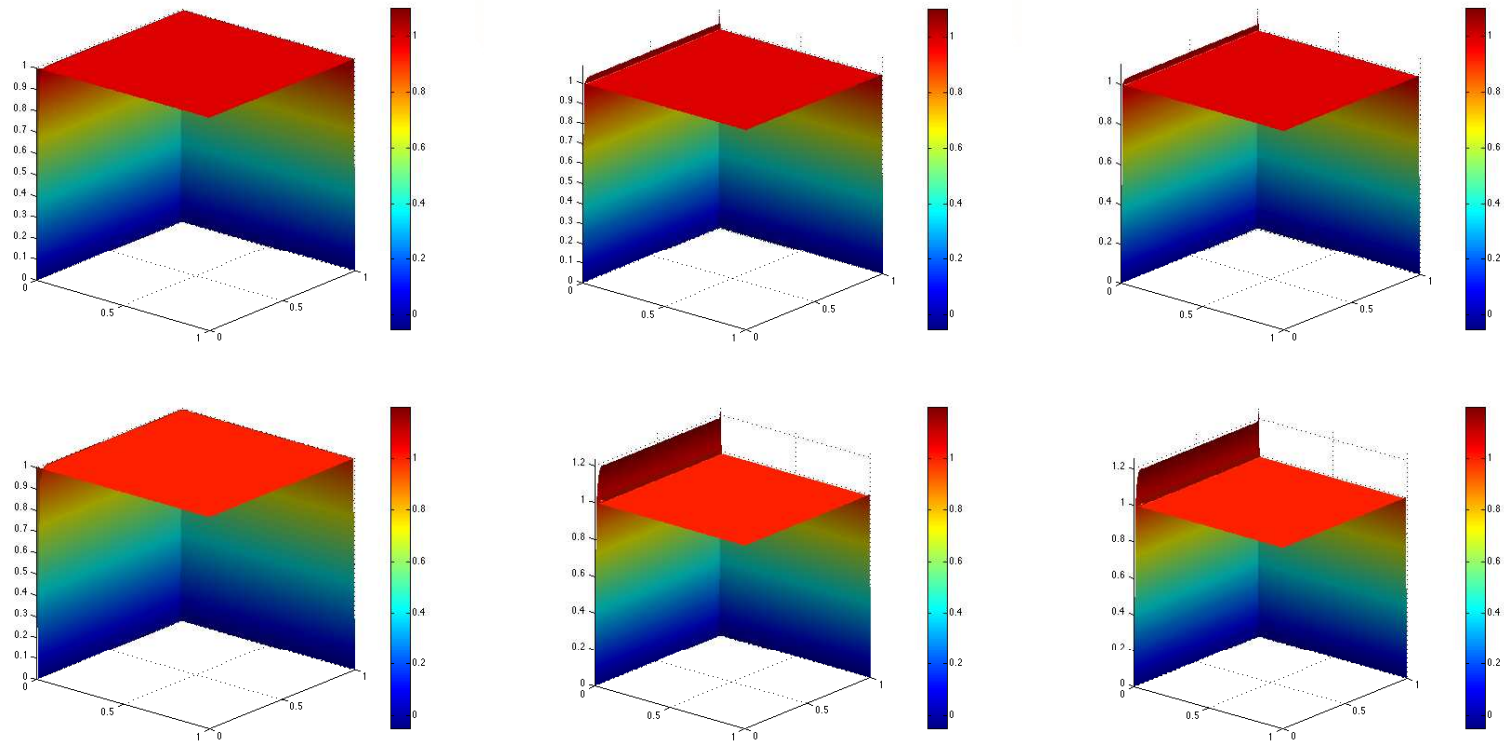
Example 1

Drift:

Moderate

Strong

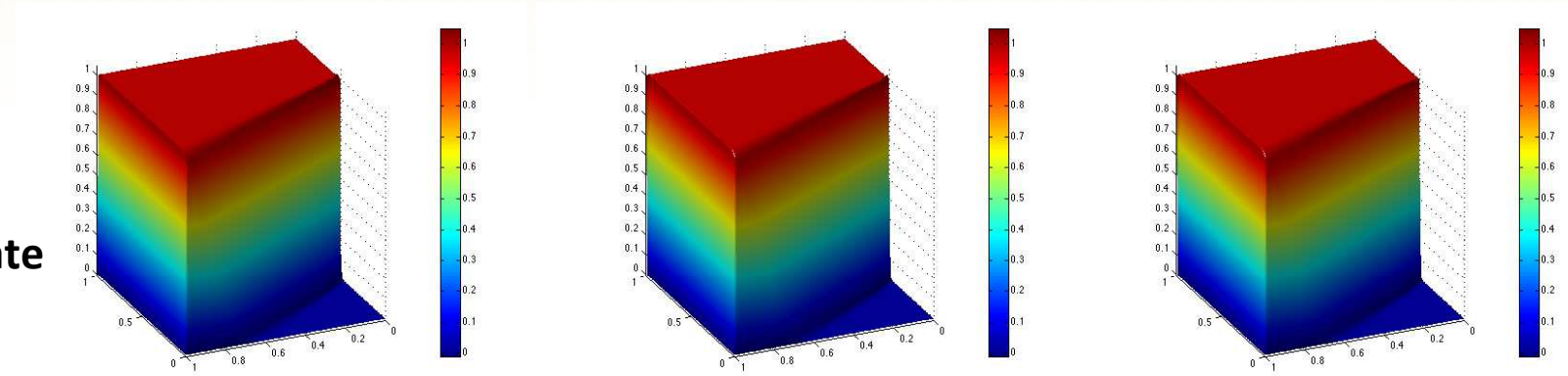


CVFEM-MS CVFEM-SU SUPG

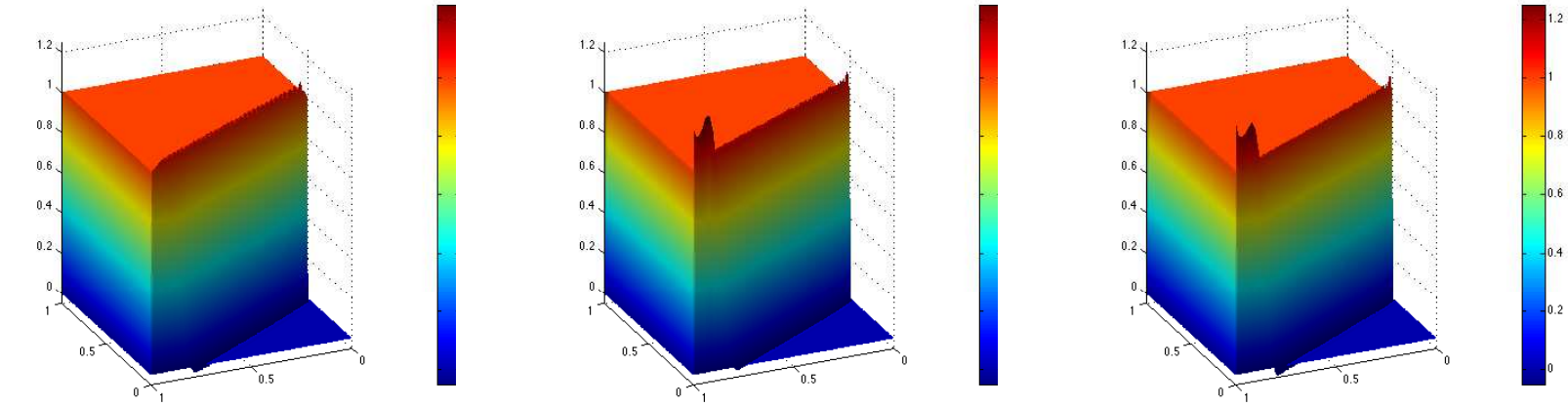
Example 2

Drift:

Moderate



Strong



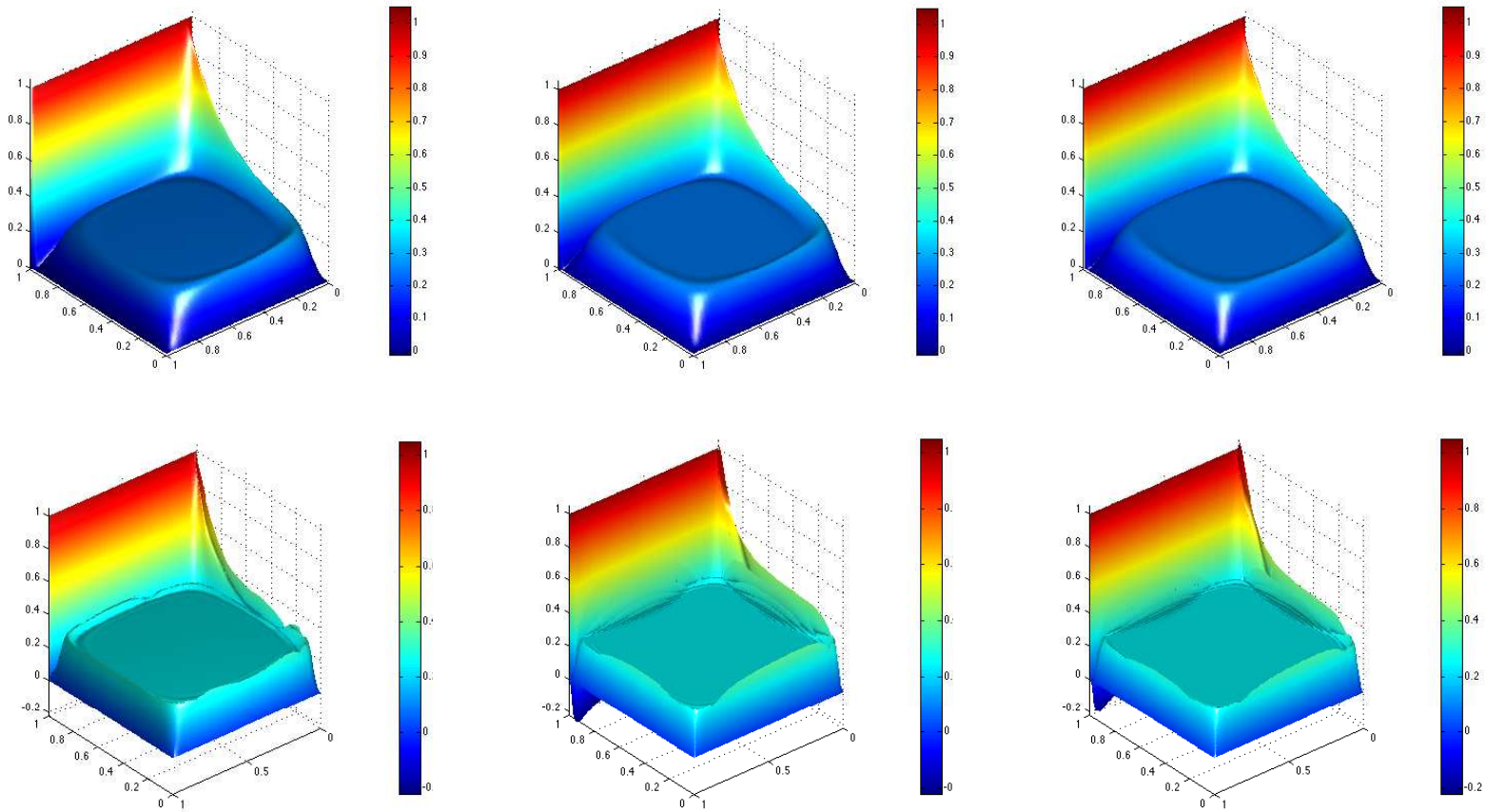
CVFEM-MS CVFEM-SU SUPG

Example 3

Drift:

Moderate

Strong



CVFEM-MS CVFEM-SU SUPG

Summary

- **The H(curl) lifting approach allows to combine the best properties of the classical Scharfetter-Gummel scheme with the flexibility of standard nodal finite elements:**
  - Locally conservative carrier current densities
  - Essentially monotone behavior (1st order formulation)
  - Topologically dual control volumes not required
  - **No tunable stabilization parameters!**

	No grid restrictions	No tunable parameters	Conservative	Essential DMP	Order
CVFEM-MS	✔	✔	✔	**Sort of …**	**2**
CVFEM-SG	✔	✔	✔	✔	**1**
CVFEM-SU	✔	✗	✔	✗	**1**
SUPG	✔	✗	✗	✗	**2**
FVM-SG	✗	✔	✔	✔	**1**

Bochev, Peterson, Gao (2013) "*A new control volume finite element method for the stable and accurate solution of the drift-diffusion equations on general unstructured grids*" CMAME 254, 126-145.

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