Order verification - facts and fiction

Pavel Bochev
Computational Mathematics and Algorithms
Sandia National Laboratories

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Definitions

Fact: a thing that is indisputably the case: Theory ⇒ Practice

Theory: under some assumptions (solution regularity, grid regularity) FE solutions should converge as \( O(h^r) \) for some \( r>0 \)

Practice: if these assumptions are satisfied, numerical solutions converge as \( O(h^r) \) too.

Fiction: a belief or statement that is false but is often held to be true because it is expedient to do so: Practice ⇒ Theory

Checking that numerical solutions converge as \( O(h^r) \) for smooth solutions on a limited set of grids (usually uniform) is sufficient to declare that the code

- is implemented correctly
- is robust and will perform well on general grids
- will recover physical solutions that are less regular than required by convergence theory, i.e., the only side effect will be slower convergence, not reduced fidelity

I will illustrate how things can go wrong using 3 codes for the Poisson (!) equation implemented using FE on quadrilateral grids.

\[
\begin{align*}
-\Delta \phi &= 0 \quad \text{in} \quad \Omega \\
\phi &= 0 \quad \text{on} \quad \Gamma
\end{align*}
\]

- Standard Galerkin method with Q1 elements
- Mixed Galerkin method with Q1-P0 elements
- Mixed least-squares method with Q1-Q1 and RT0-Q1 elements
Order verification

This is what we see on a sequence of uniform grids:

And it is exactly what the theory predicts:

\[ \| p - p^h \| = O(h^2) \quad \text{and} \quad \| u - u^h \| = O(h) \]

So the theory seems to check out just fine…
Let’s try some other grids

(A) & (B) are mild perturbations of a uniform mesh

(C) has the worst cell aspect ratio.

All grid sequences are uniformly regular
The Galerkin code seems to work fine.

Galerkin code seems to work fine.

Uniform Grid

Nodal elements have comparable performance and are optimally accurate on all 3 grids.

2x2 Gauss points used in all cases.

\[-\Delta \phi = 0 \text{ in } \Omega \]
\[\phi = 0 \text{ on } \Gamma\]
But look what happens to the Mixed Galerkin!

\[ \nabla \cdot \mathbf{v} = f \quad \text{in} \quad \Omega \\
\mathbf{v} + \nabla \phi = 0 \quad \text{in} \quad \Omega \\
\phi = 0 \quad \text{on} \quad \Gamma 
\]
What is going on here?

We were too quick to conclude that all is fine by checking the error bound

$$\| p - p^h \| = O(h) \text{ and } \| u - u^h \| = O(h)$$

only on uniform grids. It turns out that this bound does not hold on non-affine grids!

Why? The RT0 space does not contain constants \( \Rightarrow W(x) = \frac{1}{\det DF} DF \hat{W} \circ F^{-1}(x) \)

This explains our results:

In the eyeball norm, grid (C) appears worse than (A) and (B)! However, the cells of (C) are, on the average closer to affine quads, than the cells of (A) and (B)!

- Face elements can behave differently from nodal elements
- Convergence studies must be carefully designed
- Using only uniform grids can be inconclusive!
- Tests must include strongly and weakly non-affine grids
- Choice of quadrature can change convergence too!
Now let’s look at a different scenario

We take 2 different implementations of a mixed least-squares method:

- Mixed least-squares method with Q1-Q1 and RT0-Q1 elements

This is what we see on a sequence of uniform grids for a smooth solution:

<table>
<thead>
<tr>
<th>LS vs BA</th>
<th>scalar</th>
<th>vector</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H^1$</td>
<td>$H^1$</td>
</tr>
<tr>
<td>Q1-RT0</td>
<td>Q1-Q1</td>
<td></td>
</tr>
<tr>
<td>LS</td>
<td>1.00</td>
<td>1.00</td>
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<tr>
<td>BA</td>
<td>1.00</td>
<td>1.00</td>
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</tbody>
</table>

And it is exactly what the theory predicts for this method:

$$\|p - p^h\| + \|\nabla \cdot (u - u^h)\| = O(h)$$

So the theory seems to check out just fine for both methods…
Let’s try another (rough) solution

From: T. Hughes, A. Masud and J. Wan, A stabilized mixed DG method for Darcy flow

Problem is driven by Neumann boundary condition (normal flux)
Solution is physical, i.e., can be realized in an experiment
Grid has 400 uniform elements aligned with the interfaces between the strips
Recovers the physical solution!  
Recovers a spurious solution!

The trouble is, the spurious solution looks completely reasonable. Order verification did not signal any trouble for the Q1-Q1 code. Imagine this scenario in a complex multiphysics code…
This is what happens in this test:

Solution belongs to $Q_1-RT0 \Rightarrow$ recovered by $Q_1-RT0$ implementation

Least-Squares solution is a projection onto the discrete space

\[ \downarrow \]

gives the best possible approximation out of that space with respect to the energy norm

$Q_1-Q1$ Least-Squares: gives the best energy norm approximation of that solution out of $Q_1$

But $Q_1 \subset H^1$ and $H^1$ has infinite co-dimension in $H(\text{div})$!

$\Rightarrow$ Error $> \text{const.}$ $\Rightarrow Q1-Q1$ LS will never converge!

✓ Convergence for smooth solutions does not guarantee approximation of other solutions!
✓ Testing on rough but physically meaningful solutions should be part of the verification
✓ This is accepted practice in hydro codes which are expected to work for rough solutions
✓ This is seldom practiced in codes for elliptic problems despite the fact that in many cases we can only expect, e.g., $H(\text{curl})$ or $H(\text{div})$ solution (discontinuous materials, etc)