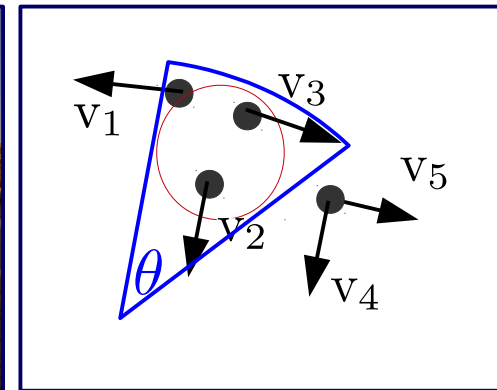
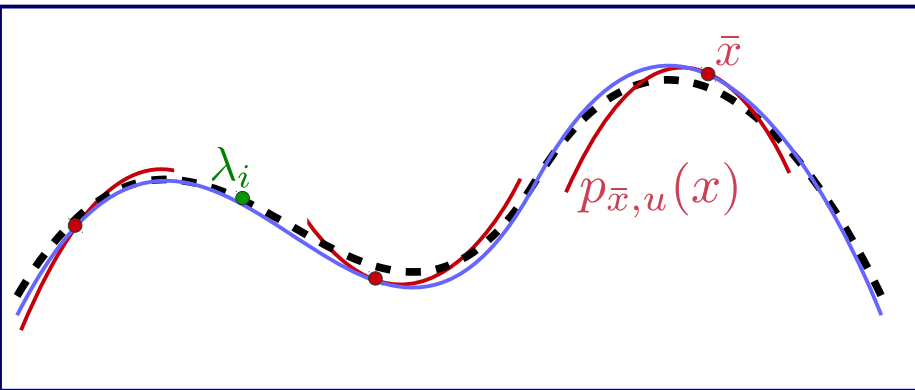


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## Advances in the Approximation Theory of Generalized Moving Least Squares

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# Generalized Moving Least Squares

**Goal:** collect and consolidate approximation results\* for GMLS

In particular we are interested in these generalizations:

- **Approximation space:** we want to consider vector polynomials and subspaces of complete polynomial spaces (e.g. div-free, curl-free spaces)
- **Sampling/target functionals:** we want to consider differential forms and in particular volume integrals, fluxes over (virtual) faces and line integral over (virtual) edges

$$\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|} \int_{e_i} \mathbf{u} \cdot \mathbf{t}_i \quad \lambda_i^f(\mathbf{u}) = \frac{1}{|f_i|} \int_{f_i} \mathbf{u} \cdot \mathbf{n}_i \quad \lambda_i^v(u) := \frac{1}{|V_i|} \int_{V_i} u(\mathbf{y}) d\mathbf{y}$$

**Applications:** *PDE discretizations and data transfer.*

In the following we present

- **generalization** of definition of “local polynomial reproduction”
- **existence and approximation results** for different sampling/target functionals

\* Mirzaei, Narcowich, Rieger, Schaback, Ward, Wendland

# Generalized Moving Least Squares

## Ingredients:

$V, V^*$  - a *function space* (e.g. **continuous functions**) and its dual

$P = \text{span}\{p_i\}_{i=1}^Q \subset V$  - a finite dimensional *approximation space*, e.g. **polynomials**

$\Lambda^h := \{\lambda_1^h, \dots, \lambda_{N_h}^h\} \subset \Lambda \subset V^*$  - a finite set of *sampling functionals* (e.g. point evaluations)  
 FE analog: *degrees of freedom*

$\tau \in \mathcal{T} \subset V^*$  - a *target functional* (or a family of target functionals)

$W(\tau, \lambda_i) : (\mathcal{T} \cup \Lambda) \times (\mathcal{T} \cup \Lambda) \rightarrow \mathbb{R}$  - a *window function* correlating functionals (e.g. a radial kernel) determines the **smoothness** of reconstruction

Example, **MLS** case:

Point cloud  $X^h = \{\mathbf{x}_i^h\}_{i=1}^{N_h} \subset \Omega$ , with filling distance  $h = \sup_{\mathbf{x}^h \in \Omega} \min_{i=1, \dots, N_h} |\mathbf{x} - \mathbf{x}_i^h|$ .

$u \in V = C^{k+1}(\Omega)$ ,  $P = \Pi^k(\Omega)$ ,  $\lambda_i^h(u) = u(\mathbf{x}_i^h)$ ,  $\tau_{\mathbf{x}} = u(\mathbf{x})$ ,  $W = W(|\mathbf{x} - \mathbf{x}_i^h|)$

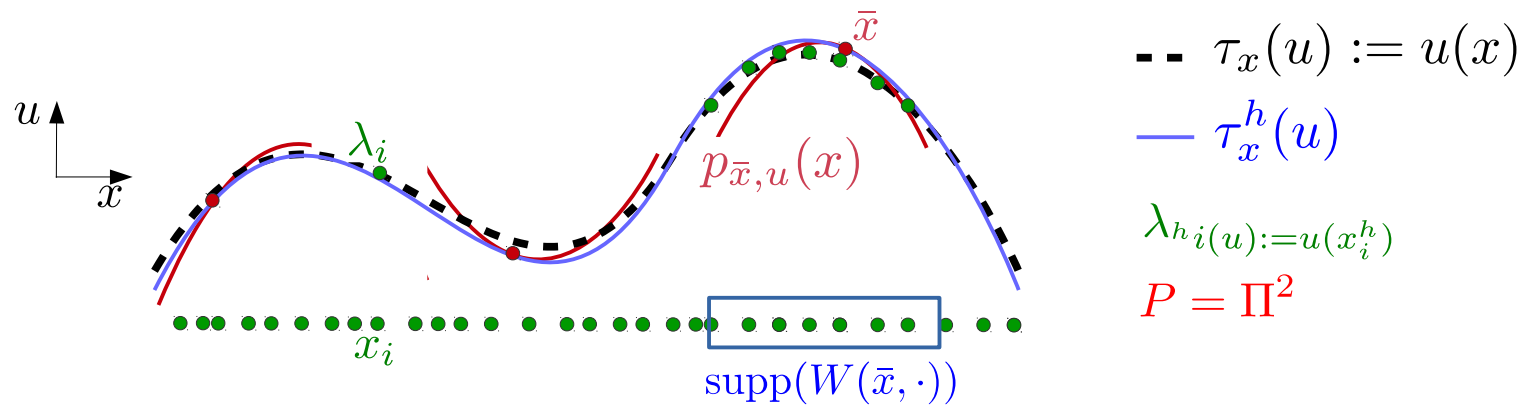
# Generalized Moving Least Squares

**Recipes** (2 equivalent formulations):

1. Least Square formulation:

$$\tau_{\bar{\mathbf{x}}}^h(u) := \tau_{\bar{\mathbf{x}}}(p_{\bar{\mathbf{x}},u}), \quad p_{\bar{\mathbf{x}},u} = \arg \min_{p \in P} \sum_{i \in I_{\bar{\mathbf{x}}}} (\lambda_i^h(u) - \lambda_i^h(p))^2 W(\bar{\mathbf{x}}, \mathbf{x}_i^h)$$

$I_{\bar{\mathbf{x}}} := \{i : W(\bar{\mathbf{x}}, \mathbf{x}_i^h) > 0\}$



2. Constrained Optimization formulation:

$$\tau_{\bar{\mathbf{x}}}^h(u) := \sum_{i \in I_{\bar{\mathbf{x}}}} a_{\tau_{\bar{\mathbf{x}}}}^i \lambda_i^h(u), \quad \{a_{\tau_{\bar{\mathbf{x}}}}^i\} = \arg \min_{a^i} \sum_{i \in I_{\bar{\mathbf{x}}}} \frac{|a^i|^2}{W(\bar{\mathbf{x}}, \mathbf{x}_i^h)},$$

s. t.  $\tau_{\bar{\mathbf{x}}}(p) = \sum_{i \in I_{\bar{\mathbf{x}}}} a^i \lambda_i^h(p), \quad \forall p \in P \quad (\tau_{\bar{\mathbf{x}}}^h(p) = \tau_{\bar{\mathbf{x}}}(p))$

# Properties of Generalized Moving Least Squares

Main questions: **When is the reconstruction possible? How good is the reconstruction?**

- **Unisolvency:**

In order for  $\tau_{\mathbf{x}}^h$  to exist,  $\Lambda_{\mathbf{x}} := \{\lambda_i\}_{i \in I_{\mathbf{x}}}$  must be  $P$ -unisolvant, i.e. for  $p \in P$

$$\forall \lambda_i \in \Lambda_{\mathbf{x}}, \lambda_i(p) = 0 \iff p = 0.$$

- **Approximation property:**

$|\tau_{\mathbf{x}}^h(u) - \tau_{\mathbf{x}}(u)| \sim O(h^s)$ , for some  $s$ .

➤ Sufficient results for unisolvancy and approx. error have been provided by Wendland\* for (non generalized) MLS.

➤ Non local sampling operators\*\*

➤ Results has been generalized by Mirzaei\*\*\*, for point derivatives as target functionals:

$|D_{\mathbf{x}}^{\alpha, h}(u) - D_{\mathbf{x}}^{\alpha}(u)| \sim O(h^{(k+1-|\alpha|)})$ , where  $k$  is the degree of the polynomials.

and for weak derivatives in Sobolev spaces\*\*\*.

\* H. Wendland, Scattered data approximation, Vol. 17. Cambridge university press, 2004

\*\* Christian Rieger, PhD thesis, personal communication.

\*\*\* D. Mirzaei et Al., IMA J. Num. Analysis, 2011, D. Mirzaei, Comp. And App. Math. 2016

## Result (**local polynomial reproduction**) [MLS]

If, for each  $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ ,  $\Lambda^h \subset \Lambda \subset V^*$ , exist  $C_{\text{loc}} > 0$  and  $C_a > 0$ , s.t.

$$\left\{ \begin{array}{l} r_{\mathbf{x}}^h(p) := \sum_i a_{\mathbf{x}}^i p(\mathbf{x}_i) = p(\mathbf{x}), \forall p \in \Pi^m \quad - \text{consistency} \\ \sum_i |a_{\mathbf{x}}^i| \leq C_a \quad - \text{uniform boundness} \\ |\mathbf{x} - \mathbf{x}_i| > C_{\text{loc}} h \Rightarrow a_{\mathbf{x}}^i = 0 \quad - \text{local support} \end{array} \right.$$

$$\implies |r_{\mathbf{x}}^h(u) - u(\mathbf{x})| \leq C_{u,m} h^{m+1}$$

## Result (**generalized local reproduction**) [GMLS]

If, for each  $\tau \in \mathcal{T} \subset V^*$ ,  $\Lambda^h \subset \Lambda \subset V^*$  there exist  $C_{\text{loc}}^h \rightarrow 0$ ,  $C_{u,\mathcal{T}}^h \rightarrow 0$ ,  $C_a^h C_{u,\Lambda}^h \rightarrow 0$ , s.t.

$$\left\{ \begin{array}{l} \tau^h(p) := \sum_i a_{\tau}^i \lambda_i(p) = \tau(p), \forall p \in P \quad - \text{consistency} \\ \sum_i |a_{\tau}^i| \leq C_a^h \quad - \text{uniform boundness} \\ d(\tau, \lambda_i) > C_{\text{loc}}^h \Rightarrow a_{\tau}^i = 0, \quad - \text{local support} \\ \forall f \in V, \exists p \in P \text{ such that:} \\ \quad |\tau(u - p)| \leq C_{u,\mathcal{T}}^h \text{ and} \quad - \text{approximation} \\ \quad |\lambda_i(u - p)| \leq C_{u,\Lambda}^h, \forall \lambda_i : d(\tau, \lambda_i) \leq C_{\text{loc}}^h. \end{array} \right.$$

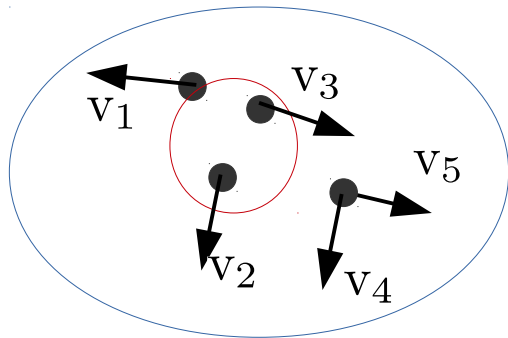
$$\implies |\tau^h(u) - \tau(u)| \leq C_{u,\mathcal{T}}^h + C_a^h C_{u,\Lambda}^h$$

# Reconstructing **vector** functions (from projections along given directions)

Associate to each particle  $i$  a position  $\mathbf{x}_i$  and a unit vector  $\mathbf{v}_i \in \mathbb{R}^d$ .

Sampling functional:  $\lambda_i(\mathbf{u}) := \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{v}_i$

Filling distance  $h_\omega$  is the radius of the smallest ball that centered at any point of  $\Omega$  contains  $d$  particles whose versors  $\mathbf{v}_1, \dots, \mathbf{v}_d$  contain a basis for  $\mathbb{R}^d$  with associated determinant bigger than  $\omega$  ( $|\det[\mathbf{v}_1, \dots, \mathbf{v}_d]| \geq \omega$ ).



For  $h_\omega \leq C(\theta, R, m, \omega)$ ,  $\forall p \in [\Pi^m]^d$ ,  
 $\exists \lambda_{i_p} \in \Lambda_h$ , such that  $|\lambda_{i_p}(p)| \geq \rho_\omega \|p\|_{\Omega, \infty}$

Other sampling functionals we considered:

$$\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|} \int_{e_i} \mathbf{u} \cdot \mathbf{t}_i$$

$$\lambda_i^f(\mathbf{u}) = \frac{1}{|f_i|} \int_{f_i} \mathbf{u} \cdot \mathbf{n}_i$$

$$\lambda_i^v(u) := \frac{1}{|V_i|} \int_{V_i} u(\mathbf{y}) d\mathbf{y}$$

# GMLS approximation results:

Basic technique:

$$\begin{aligned}
 |\tau(u) - \tau^h(u)| &\leq |\tau(u) - \tau(p)| + |\tau(p) - \tau^h(u)|, && (\forall p \in P) \\
 &\leq |\tau(u) - \tau(p)| + |\tau^h(p - u)|, && \longleftarrow \text{reconstruction property} \\
 &\leq |\tau(u - p)| + \left| \sum_{i=1}^{N_p} \lambda_i(u - p) a_\tau^i \right| && \longleftarrow \text{GMLS definition} \\
 &\leq |\tau(u - p)| + \max_{i \in I_\tau} |\lambda_i(u - p)| \sum_{i \in I_\tau} |a_\tau^i|.
 \end{aligned}$$

$$\sum_{i \in I_\tau} |a_\tau^i| \leq C_W \|\tau\|_{P^*} \|\Lambda^{-1}\|$$

Holds for any target functional and approximation space:

$$|\tau(u) - \tau^h(u)| \leq |\tau(u - p)| + C_W \|\tau\|_{P^*} \|\Lambda^{-1}\| \max_{i \in I_\tau} |\lambda_i(u - p)|, \quad p \in P$$



# GMLS approximation results:

## integral target functionals (e.g. used for nonlocal problems)

$$\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|} \int_{e_i} \mathbf{u} \cdot \mathbf{t}_i, \quad V = C^{k+1}(\Omega), \quad P = [\Pi^k(\Omega)]^d$$

$$\tau_{\mathbf{x}}(u) = \text{div}(\mathbf{u})|_{\mathbf{x}}$$

$$|\lambda_i^e(\mathbf{u})| \leq \|\mathbf{u}\|_{L^\infty(e_i)}$$

Take  $p = p_{\mathbf{x},u}^k$ , the Taylor polynomial of degree  $k$  of  $u$  at  $\mathbf{x}$ .

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau_{\mathbf{x}}^h(\mathbf{u})| \leq |\tau_{\mathbf{x}}(\mathbf{u} - p_{\mathbf{x},u}^k)| + C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i \in I_{\mathbf{x}}} |\lambda_i(\mathbf{u} - p_{\mathbf{x},u}^k)|$$

Taylor approx.  
( $k \geq 1$ )



Mirzaei\*\*

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau_{\mathbf{x}}^h(\mathbf{u})| \leq 0 + Ch^{-1} h^{k+1} \|\mathbf{u}\|_{\infty, \Omega}, \quad \text{if } |e_i| \sim h$$

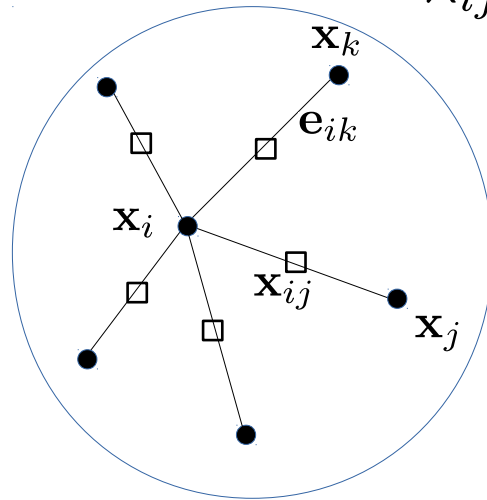


$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau_{\mathbf{x}}^h(\mathbf{u})| \leq Ch^k \|\mathbf{u}\|_{\infty, \Omega}$$

\*\* D. Mirzaei et Al., IMA J. Num. Analysis, 2011.

# Application to PDE: staggered scheme for Darcy

$$\begin{cases} \operatorname{div}(\mathbf{u}) = \mathbf{0} \\ \mathbf{u} = -K \nabla \phi \end{cases}$$



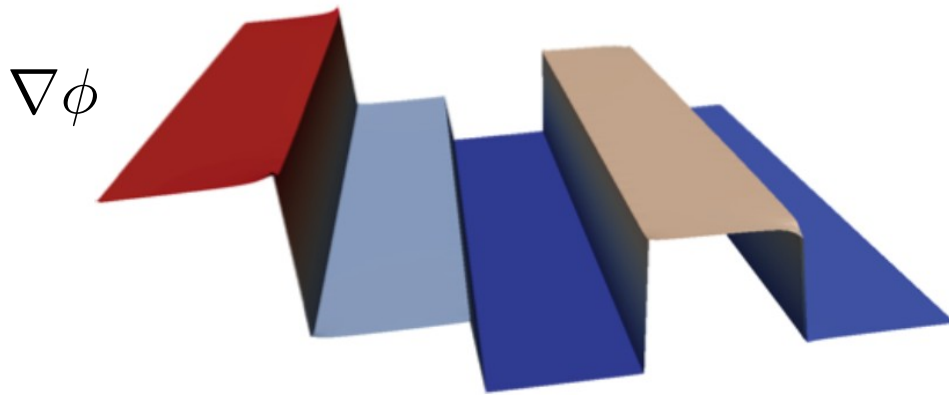
$$\lambda_{ij}(\mathbf{u}) := \int_{e_{ij}} \mathbf{u} \cdot \mathbf{t}_{ij} \approx K(\mathbf{x}_{ij})(\phi_j - \phi_i)$$

$$\tau_{\mathbf{x}_i}(\mathbf{u}) = \operatorname{div}(\mathbf{u})|_{\mathbf{x}_i}$$

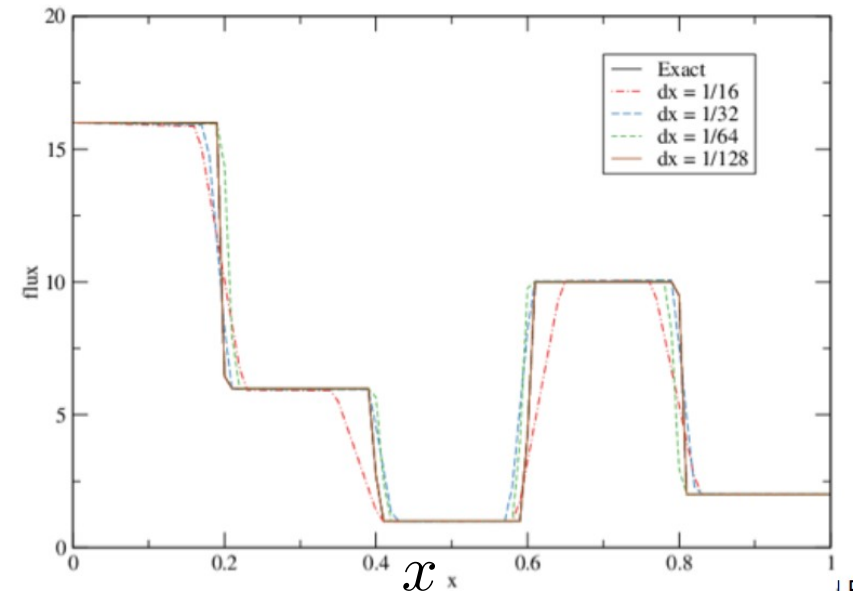
$$P = [\Pi_m]^d$$

$$W(\tau_{\mathbf{x}}, \lambda_{ij}) = W(|\mathbf{x} - \mathbf{x}_{ij}|)$$

Solution of PDE w/ jumps in permeability



$\nabla \phi$



Thank you!

$$\tau \in \mathcal{T} \subset V^*, \quad \Lambda^h = \{\lambda_i^h \in V^*\}_{i=1}^{N_h} \subset \Lambda \subset V^*$$

Pseudo-distance:

$$\mathcal{X} : (\mathcal{T} \cup \Lambda) \rightarrow \mathbb{R}^q \quad d^*(\tau, \lambda) := |\mathcal{X}(\tau) - \mathcal{X}(\lambda)|$$

Separation and filling distances:

$$h_s = \frac{1}{2} \min_{i \neq j} d^*(\lambda_i, \lambda_j), \quad h = \sup_{\tau \in \mathcal{T}} \min_i d^*(\tau, \lambda_i),$$

Quasi-uniformity:

$$h_s \leq h \leq C_{qu} h_s$$

Properties of window function  $W : \mathcal{T} \times \Lambda \rightarrow \mathbb{R}$ :

$$0 \leq W(\tau, \lambda) \leq M,$$

$$W(\tau, \lambda) = 0 \quad \text{if } d^*(\tau, \lambda) > C_{loc}^h$$

$$W(\tau, \lambda) \geq \rho > 0 \quad \text{if } d^*(\tau, \lambda) < \frac{1}{2} C_{loc}^h$$

Neighborhood:

$$I_\tau := \{i, | W(\tau, \lambda_i) > 0\}, \quad \bar{I}_\tau := \{i, | d^*(\tau, \lambda_i) < \frac{1}{2} C_h\}$$

Sample mapping

$$T_\tau : P \rightarrow \mathbb{R}^{\#\tilde{I}_\tau}, \quad T_\tau(v) := (\lambda_i(v))_{i \in \tilde{I}_\tau}$$

**GMLS process defines a local reproduction if:**

$$\forall \Lambda_h, h \leq h_0 :$$

$$\|\tau\|_{V^*} \leq C_U h^{\beta_1}, \quad \|\lambda_i\|_{V^*} \leq C_\Lambda h^{\beta_2}$$

$$\|\tilde{T}_\tau^{-1}\| \leq C_\Lambda h^{\beta_3}$$

$\forall f \in V, \exists p \in P$  such that:

$$\|f - p\|_{V \setminus \text{Ker}(\tau)} \leq C_f h^{\alpha_1} \text{ and}$$

$$\|f - p\|_{V \setminus \text{Ker}(\lambda_i)} \leq C_f h^{\alpha_2}, \quad \forall \lambda_i : d(\tau, \lambda_i) \leq C_h.$$