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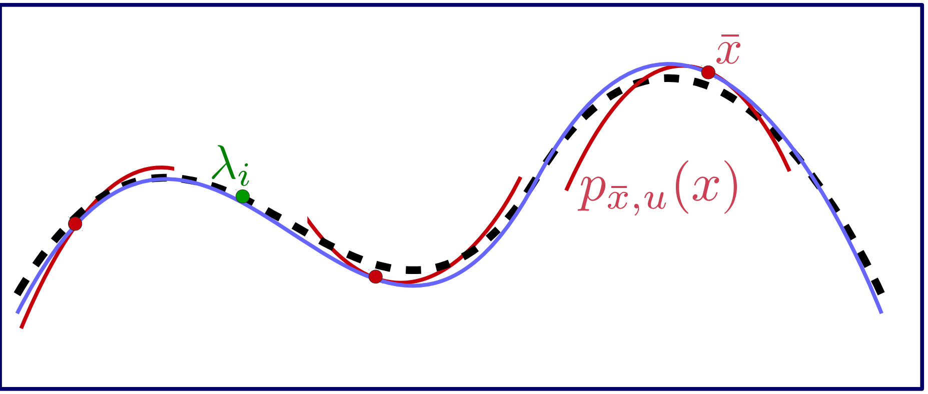

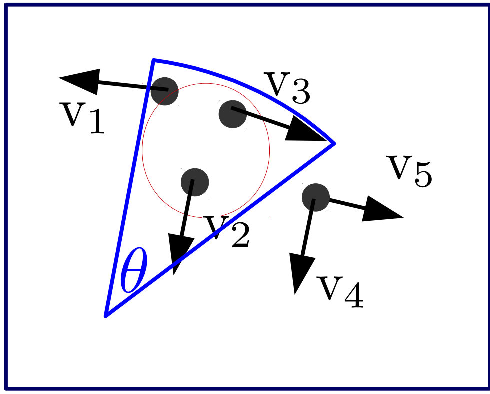

Advances in the Approximation Theory for Generalized Moving Least Squares

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Generalized Moving Least Squares

Ingredients:

V, V^* - a *function space* (e.g. continuous functions) and its dual

$P = span\{p_i\}_{i=1}^{Q} \subset V$ - a finite dimensional *approximation space*, e.g. polynomials
FE analog: *local approximation space within an element*

$\Lambda^h := \{\lambda_1^h, \ldots, \lambda_{N_h}^h\} \subset \Lambda \subset V^*$ - a finite set of *sampling functionals* (e.g. point evaluations)
FE analog: *degrees of freedom*

$\tau \in \mathcal{T} \subset V^*$ - a *target functional* (or a family of target functionals)

$W(\tau, \lambda_i) : (\mathcal{T} \cup \Lambda) \times (\mathcal{T} \cup \Lambda) \to \mathbb{R}$ - a *window function* correlating functionals (e.g. a radial kernel) determines the **smoothness** of reconstruction
FE analog: 1 if target and sampling functionals belong to the same element, 0 otherwise.

Example, MLS case:

Point cloud $X^h = \{\mathbf{x}_i^h\}_{i=1}^{N_h} \subset \Omega$, with filling distance $h = \sup_{\mathbf{x}^h \in \Omega} \min_{i=1,\ldots,N_h} |\mathbf{x} - \mathbf{x}_i|$.

$$u \in V = C^{k+1}(\Omega), \quad P = \Pi^k(\Omega), \quad \lambda_i^h(u) = u(\mathbf{x}_i^h), \quad \tau_{\mathbf{x}} = u(\mathbf{x}), \quad W = W(|\mathbf{x} - \mathbf{x}_i^h|)$$

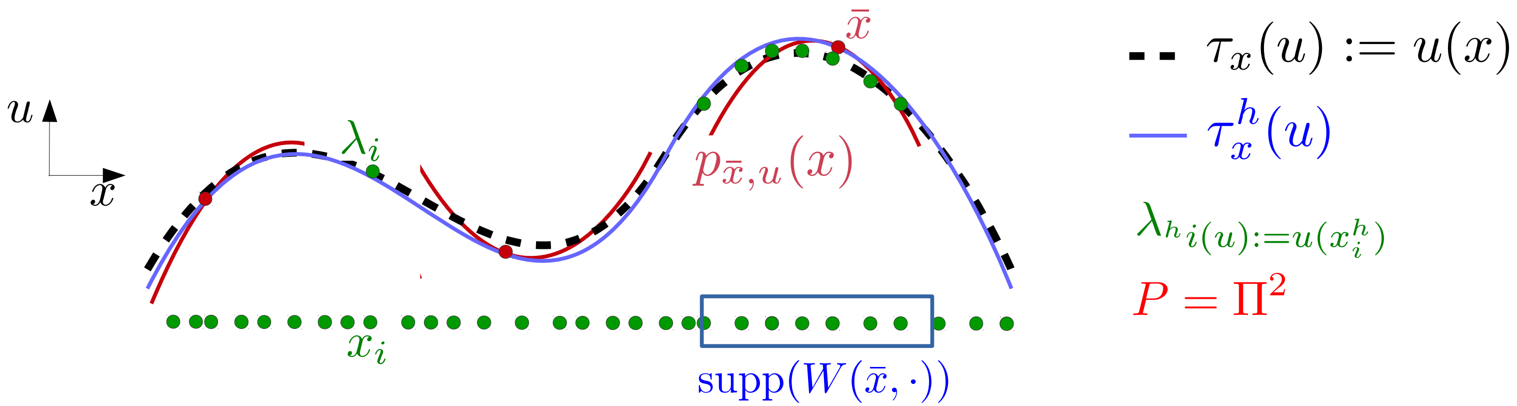
Generalized Moving Least Squares

Recipes (2 equivalent formulations):

1. Least Square formulation:

$$\boxed{\tau_{\bar{\mathbf{x}}}^h(u) := \tau_{\bar{\mathbf{x}}}(p_{\bar{\mathbf{x}},u}),} \qquad p_{\bar{\mathbf{x}},u} = \arg\min_{p\in P} \sum_{i\in I_{\bar{\mathbf{x}}}} \left(\lambda_i^h(u) - \lambda_i^h(p)\right)^2 W(\bar{\mathbf{x}}, \mathbf{x}_i^h)$$

$\mathrm{I}_{\bar{\mathbf{x}}} := \{i : W(\bar{\mathbf{x}}, \mathbf{x}_i^h) > 0\}$



- - $\tau_x(u) := u(x)$

— $\tau_x^h(u)$

$\lambda_{{}^h i(u):=u(x_i^h)}$

$P = \Pi^2$

2. Constrained Optimization formulation:

$$\boxed{\tau_{\bar{\mathbf{x}}}^h(u) := \sum_{i\in I_{\bar{\mathbf{x}}}} a^i_{\tau_{\bar{\mathbf{x}}}} \lambda_i^h(u),} \qquad \{a^i_{\tau_{\mathbf{x}}}\} = \arg\min_{a^i} \sum_{i\in I_{\bar{\mathbf{x}}}} \frac{|a^i|^2}{W(\bar{\mathbf{x}}, \mathbf{x}_i^h)},$$

$$\text{s. t. } \tau_{\bar{\mathbf{x}}}(p) = \sum_{i\in I_{\bar{\mathbf{x}}}} a^i \lambda_i^h(p),\ \forall p \in P \quad \left(\tau_{\bar{\mathbf{x}}}^h(p) = \tau_{\bar{\mathbf{x}}}(p)\right)$$

Properties of Generalized Moving Least Squares

Main questions: **When is the reconstruction possible? How good is the reconstruction?**

- **Unisolvency**:
  In order for $\tau_{\mathbf{x}}^h$ to exist, $\Lambda_{\mathbf{x}} := \{\lambda_i\}_{i \in I_{\mathbf{x}}}$ must be P-unisolvent, i.e. for $p \in P$

$$\forall \lambda_i \in \Lambda_{\mathbf{x}},\ \lambda_i(p) = 0 \iff p = 0.$$

FE DoFs are unisolvent by construction

- **Approximation property**:

$$|\tau_{\mathbf{x}}^h(u) - \tau_{\mathbf{x}}(u)| \sim O(h^s),\ \text{for some } s.$$

➢ Sufficient results for unisolvancy and approx. error have been provided by Wendland* for (non generalized) MLS.

➢ Results has been generalized by Mirzaei**, for point derivatives as target functionals:

$$|D_{\mathbf{x}}^{\alpha,h}(u) - D_{\mathbf{x}}^{\alpha}(u)| \sim O(h^{(k+1-|\alpha|)}),\ \text{where } k \text{ is the degree of the polynomials.}$$

and for weak derivatives in Sobolev spaces***.

* H. Wendland, Scattered data approximation, Vol. 17. Cambridge university press, 2004.
** D. Mirzaei et Al., IMA J. Num. Analysis, 2011.
*** D. Mirzaei, Comp. And App. Math. 2016

Generalized Moving Least Squares: our plan

Our plan is to further extend analysis of generalized moving least square, and present **general approximation results**.

- **Sampling/target functionals**: we want to consider differential forms and in particular volume integrals, fluxes over (virtual) faces and line integral over (virtual) edges. Potentially useful for ***PDE discretizations and data transfer***.

- **Approximation space**: we want to consider vector polynomials and subspaces of complete polynomial spaces (e.g. div-free, curl-free spaces).

- **Go beyond smooth functions** and consider functions in Sobolev spaces (expanding on what done in Mirzaei ***)

In the following we present
1) **existence results** for different sampling functionals
2) **approximation results** for different sampling/target functionals
3) **generalization** of definition of "local polynomial reproduction"

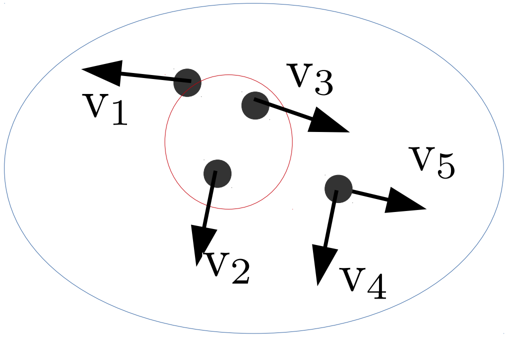
*** D. Mirzaei, Comp. And App. Math. 2016

Reconstructing **vector** functions
(from projections along given directions)

Associate to each particle i a position $\mathbf{x}_i$ and a unit vector $\mathbf{v}_i \in \mathbb{R}^d$.

Sampling functional: $\boxed{\lambda_i(\mathbf{u}) := \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{v}_i}$

Filling distance h_ω is the radius of the smallest ball that centered at any point of Ω contains d particles whose versors $\mathbf{v}_1, \dots, \mathbf{v}_d$ contain a basis for $\mathbb{R}^d$ with associated determinant bigger than ω ($|\,\mathbf{det}[\mathbf{v_1}, \dots, \mathbf{v}_{\boldsymbol{d}}]| \geq \boldsymbol{w}$).



$$\boxed{\begin{array}{l}\text{For } h_\omega \leq C(\theta, R, m, w),\ \forall p \in [\Pi^m]^d,\\ \exists\, \lambda_{i_p} \in \Lambda_h,\ \text{such that } |\lambda_{i_p}(p)| \geq \rho_\omega \|p\|_{\Omega,\infty}\end{array}}$$

Other sampling functionals we considered:

$$\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|}\int_{e_i} \mathbf{u} \cdot \mathbf{t}_i \qquad \lambda_i^f(\mathbf{u}) = \frac{1}{|f_i|}\int_{f_i} \mathbf{u} \cdot \mathbf{n}_i \qquad \lambda_i^v(u) := \frac{1}{|V_i|}\int_{V_i} u(\boldsymbol{y})\, d\boldsymbol{y}$$

GMLS approximation results:

Basic technique:

$$\begin{aligned}
|\tau_{\mathbf{x}}(u) - \tau^h_{\mathbf{x}}(u)| \quad &\leq |\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}(p)| + |\tau_{\mathbf{x}}(p) - \tau^h_{\mathbf{x}}(u)|, \qquad (\forall p \in P) \\
&\leq |\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}(p)| + |\tau^h_{\mathbf{x}}(p-u)|, \quad \longleftarrow \text{ reconstruction property} \\
&\leq |\tau_{\mathbf{x}}(u-p)| + \left| \sum_{i=1}^{N_p} \lambda_i(u-p)\, a^i_{\tau_{\mathbf{x}}} \right| \quad \longleftarrow \text{ GMLS definition} \\
&\leq |\tau_{\mathbf{x}}(u-p)| + \max_{i \in I_{\mathbf{x}}} |\lambda_i(u-p)| \sum_{i \in I_{\mathbf{x}}} |a^i_{\tau_{\mathbf{x}}}|.
\end{aligned}$$

$$\boxed{\sum_{i \in I_{\mathbf{x}}} |a^i_{\tau_{\mathbf{x}}}| \leq C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda^{-1}_{\mathbf{x}}\|}$$

Holds for any target functional and approximation space:

$$\boxed{|\tau_{\mathbf{x}}(u) - \tau^h_{\mathbf{x}}(u)| \leq |\tau_{\mathbf{x}}(u-p)| + C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda^{-1}_{\mathbf{x}}\| \max_{i \in I_{\mathbf{x}}} |\lambda_i(u-p)|, \qquad p \in P}$$

GMLS approximation results: **div free** spaces

$$V = \{\mathbf{u} \in [C^{k+1}(\Omega)]^d, \mathrm{div}(\mathbf{u}) = 0\}, \quad P = \{\mathbf{p} \in [\Pi^k(\Omega)]^d, \mathrm{div}(\mathbf{p}) = 0\}$$

$$\lambda_i^j(\mathbf{u}) := \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{e}_j, \quad \tau_{\mathbf{x}}^j(\mathbf{u}) = \mathbf{u}(\mathbf{x}) \cdot \mathbf{e}_j, \quad \boldsymbol{\tau}_{\mathbf{x}} = (\tau_{\mathbf{x}}^1, \ldots, \tau_{\mathbf{x}}^d)$$

The Taylor polynomial $\mathbf{p}_{\mathbf{x},u}^k$ of $\mathbf{u} \in V$, belongs to P (it's divergence free).

$$|\tau_{\mathbf{x}}^j(\mathbf{u}) - \tau_{\mathbf{x}}^{j,h}(\mathbf{u})| \leq |\tau_{\mathbf{x}}^j(\mathbf{u} - \mathbf{p}_{\mathbf{x},u}^k)| + C_W \|\tau_{\mathbf{x}}^j\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i \in I_{\mathbf{x}}} |\lambda_i^j(\mathbf{u} - \mathbf{p}_{\mathbf{x},\mathbf{u}}^k)|$$

Taylor approx. ($|\alpha| \leq k$) $\downarrow$ 0

Taylor approx. $\downarrow$ $\mathcal{O}\left(h^{k+1}\right)$

$$\boxed{|\boldsymbol{\tau}_{\mathbf{x}}(\mathbf{u}) - \boldsymbol{\tau}_{\mathbf{x}}^h(\mathbf{u})| \sim \mathcal{O}(h^{k+1})}$$

Note that, in general, $\mathrm{div}(\tau_{\mathbf{x}}^h(\mathbf{u})) \neq 0$.

GMLS approximation results:
integral target functionals (e.g. used for nonlocal problems)

$$\boxed{\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|}\int_{e_i} \mathbf{u}\cdot\mathbf{t}_i,} \qquad V = C^{k+1}(\Omega), \qquad P = \left[\Pi^k(\Omega)\right]^d$$

$$\boxed{\tau_{\mathbf{x}}(u) = \mathrm{div}(\mathbf{u})|_{\mathbf{x}}} \qquad |\lambda_i^e(\mathbf{u})| \leq \|\mathbf{u}\|_{L^\infty(e_i)}$$

Take $p = p^k_{\mathbf{x},u}$, the Taylor polynomial of degree k of u at $\mathbf{x}$.

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau^h_{\mathbf{x}}(\mathbf{u})| \leq |\tau_{\mathbf{x}}(\mathbf{u} - p^k_{\mathbf{x},u})| + C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i\in I_{\mathbf{x}}} |\lambda_i(\mathbf{u} - p^k_{\mathbf{x},u})|$$

Taylor approx. $(k \geq 1)$ ↓ $\quad\Downarrow\quad$ ↓ Mirzaei**

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau^h_{\mathbf{x}}(\mathbf{u})| \leq \quad 0 \quad + \quad Ch^{-1}\, h^{k+1} \|\mathbf{u}\|_{\infty,\Omega}, \qquad \text{if } |e_i| \sim h$$

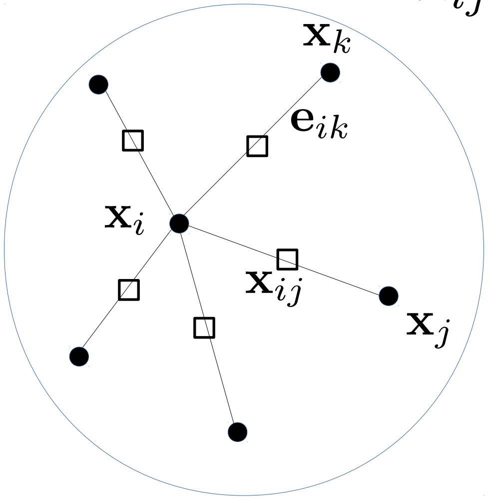
$$\Downarrow$$

$$\boxed{|\tau_{\mathbf{x}}(\mathbf{u}) - \tau^h_{\mathbf{x}}(\mathbf{u})| \leq Ch^k \|\mathbf{u}\|_{\infty,\Omega}}$$

** D. Mirzaei et Al., IMA J. Num. Analysis, 2011.

Application to PDE: staggered scheme for Darcy

$$\left\{\begin{array}{l}\operatorname{div}(\mathbf{u}) = \mathbf{0}\\ \mathbf{u} = -K\nabla\phi\end{array}\right.$$



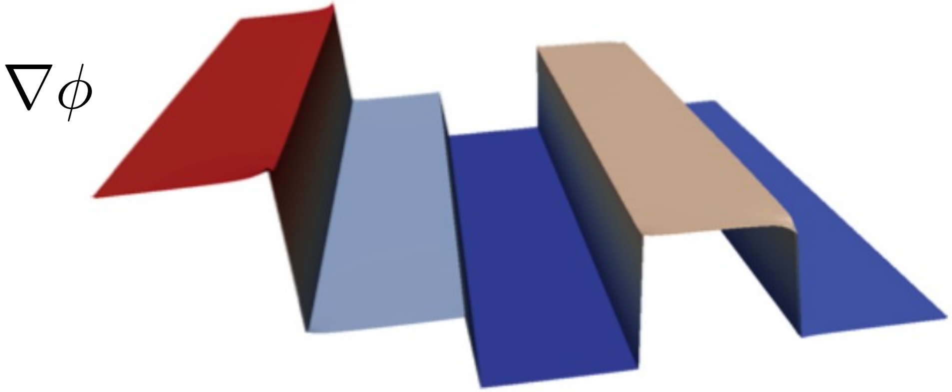
$$\lambda_{ij}(\mathbf{u}) := \int_{\mathbf{e}_{ij}} \mathbf{u}\cdot\mathbf{t}_{ij} \approx K(\mathbf{x}_{ij})(\phi_j - \phi_i)$$

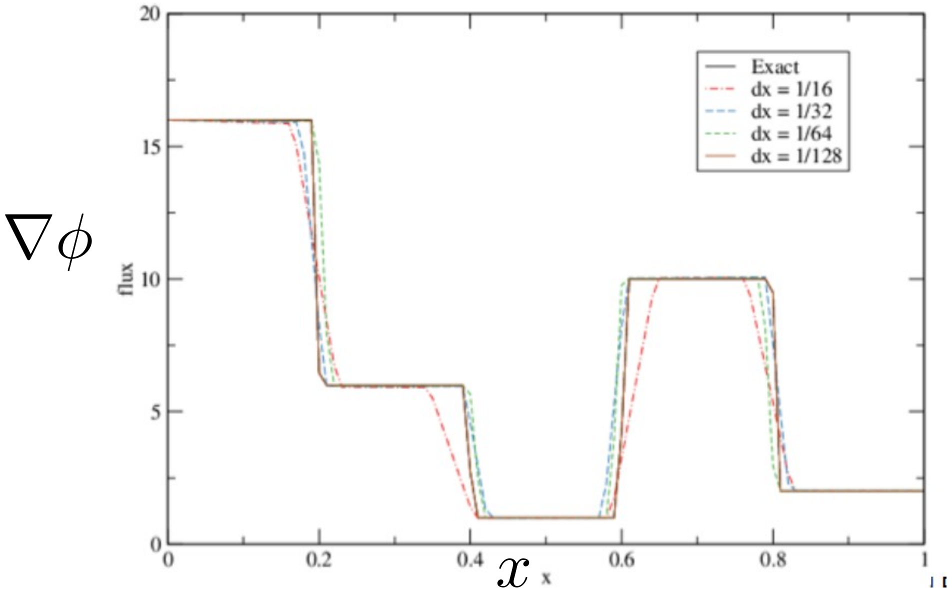
$$\tau_{\mathbf{x}_i}(\mathbf{u}) = \operatorname{div}(\mathbf{u})|_{\mathbf{x}_i}$$

$$P = [\Pi_m]^d$$

$$W(\tau_{\mathbf{x}}, \lambda_{ij}) = W(|\mathbf{x} - \mathbf{x}_{ij}|)$$

Solution of PDE w/ jumps in permeability





Towards a general theory for GMLS: local reproduction

Result (**local polynomial reproduction**) [MLS]

If, for each $\mathbf{x} \in \Omega \subset \mathbb{R}^d$, $\Lambda^h \subset \Lambda \subset V^*$, exist $C_{\mathrm{loc}} > 0$ and $C_a > 0$, s.t.

$$\begin{cases} r_{\mathbf{x}}^h(p) := \sum_i a_{\mathbf{x}}^i \, p(\mathbf{x}_i) = p(\mathbf{x}), \; \forall p \in \Pi^m & \text{- consistency} \\ \sum_i |a_{\mathbf{x}}^i| \le C_a & \text{- uniform boundness} \\ |\mathbf{x} - \mathbf{x}_i| > C_{\mathrm{loc}} h \;\Rightarrow\; a_{\mathbf{x}}^i = 0 & \text{- local support} \end{cases}$$

$$\Longrightarrow \; |r_{\mathbf{x}}^h(u) - u(\mathbf{x})| \le C_{u,m} h^{m+1}$$

Result (**generalized local reproduction**) [GMLS]

If, for each $\tau \in \mathcal{T} \subset V^*$, $\Lambda^h \subset \Lambda \subset V^*$ there exist $C^h_{\mathrm{loc}} \to 0$, $C^h_{u,\mathcal{T}} \to 0$, $C^h_a C^h_{u,\Lambda} \to 0$, s.t.

$$\begin{cases} \tau^h(p) := \sum_i a_\tau^i \lambda_i(p) = \tau(p), \; \forall p \in P & \text{- consistency} \\ \sum_i |a_\tau^i| \le C_a^h & \text{- uniform boundness} \\ d(\tau, \lambda_i) > C^h_{\mathrm{loc}} \;\Rightarrow\; a_\tau^i = 0, & \text{- local support} \\ \forall f \in V, \exists p \in P \text{ such that:} & \\ \quad |\tau(u-p)| \le C^h_{u,\mathcal{T}} \text{ and} & \text{- approximation} \\ \quad |\lambda_i(u-p)| \le C^h_{u,\Lambda}, \; \forall \lambda_i : d(\tau,\lambda_i) \le C^h_{\mathrm{loc}}. & \end{cases}$$

$$\Longrightarrow \; |\tau^h(u) - \tau(u)| \le C^h_{u,\mathcal{T}} + C^h_a C^h_{u,\Lambda}$$

Thank you!

Towards a general theory for GMLS

$$\tau \in \mathcal{T} \subset V^*, \quad \Lambda^h = \{\lambda_i^h \in V^*\}_{i=1}^{N_h} \subset \Lambda \subset V^*$$

Psudo-distance:

$$\mathcal{X} : (\mathcal{T} \cup \Lambda) \to \mathbb{R}^q \qquad d^*(\tau, \lambda) := |\mathcal{X}(\tau) - \mathcal{X}(\lambda)|$$

Separation and *filling* distances:

$$h_s = \frac{1}{2} \min_{i \neq j} d^*(\lambda_i, \lambda_j), \qquad h = \sup_{\tau \in \mathcal{T}} \min_i d^*(\tau, \lambda_i),$$

Quasi-uniformity:

$$h_s \leq h \leq C_{qu} h_s$$

Properties of window function $W : \mathcal{T} \times \Lambda \to \mathbb{R}$:

$$\begin{aligned} & 0 \leq W(\tau, \lambda) \leq M, \\ & W(\tau, \lambda) = 0 && \text{if } d^*(\tau, \lambda) > C_{\text{loc}}^h \\ & W(\tau, \lambda) \geq \rho > 0 && \text{if } d^*(\tau, \lambda) < \tfrac{1}{2} C_{\text{loc}}^h \end{aligned}$$

Neighborhood:

$$I_\tau := \{i, \,|\, W(\tau, \lambda_i) > 0\}, \quad \bar{I}_\tau := \{i, \,|\, d^*(\tau, \lambda_i) < \frac{1}{2} C_h\}$$

Towards a general theory for GMLS

Sample mapping

$$T_\tau : P \to \mathbb{R}^{\#\tilde{I}_\tau}, \quad T_\tau(v) := \big(\lambda_i(v)\big)_{i\in\tilde{I}_\tau}$$

GMLS process defines a local reproduction if:

$$\forall \Lambda_h,\, h \le h_0 :$$

$$\|\tau\|_{V^*} \le C_U h^{\beta_1}, \quad \|\lambda_i\|_{V^*} \le C_\Lambda h^{\beta_2}$$

$$\|\tilde{T}_\tau^{-1}\| \le C_\Lambda h^{\beta_3}$$

$$\forall f \in V, \exists p \in P \text{ such that:}$$

$$\|f-p\|_{V\backslash \mathrm{Ker}(\tau)} \le C_f h^{\alpha_1} \text{ and}$$

$$\|f-p\|_{V\backslash \mathrm{Ker}(\lambda_i)} \le C_f h^{\alpha_2},\ \forall \lambda_i : d(\tau,\lambda_i) \le C_h.$$