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Domain Decomposition Methods for Advection Dominated Linear–Quadratic Elliptic Optimal Control Problems *

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Abstract

We present an optimization-level domain decomposition (DD) preconditioner for the solution of advection dominated elliptic linear–quadratic optimal control problems, which arise in many science and engineering applications. The DD preconditioner is based on a decomposition of the optimality conditions for the elliptic linear–quadratic optimal control problem into smaller subdomain optimality conditions with Dirichlet boundary conditions for the states and the adjoints on the subdomain interfaces. These subdomain optimality conditions are coupled through Robin transmission conditions for the states and the adjoints. The parameters in the Robin transmission condition depend on the advection. This decomposition leads to a Schur complement system in which the unknowns are the state and adjoint variables on the subdomain interfaces. The Schur complement operator is the sum of subdomain Schur complement operators, the application of which is shown to correspond to the solution of subdomain optimal control problems, which are essentially smaller copies of the original optimal control problem. We show that, under suitable conditions, the application of the inverse of the subdomain Schur complement operators requires the solution of a subdomain elliptic linear–quadratic optimal control problem with Robin boundary conditions for the state.

Numerical tests for problems with distributed and with boundary control show that the dependence of the preconditioners on mesh size and subdomain size is comparable to its counterpart applied to a single advection dominated equation. These tests also show that the preconditioners are insensitive to the size of the control regularization parameter.

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1 Introduction

Optimization problems governed by (systems of) advection dominated elliptic partial differential equations (PDEs) arise in many science and engineering applications, see, e.g., [1, 2, 7, 8, 15, 16, 17, 19, 23, 32, 35], either directly or as subproblems in Newton-type or sequential quadratic optimization algorithms for the solution of optimization problems governed by (systems of) nonlinear PDEs. This paper is concerned with optimization–level domain decomposition preconditioners for such problems. We focus our presentation on the linear quadratic optimal control problem

$$\text{minimize } \frac{1}{2}\int_\Omega (y(x)-\widehat{y}(x))^2 dx + \frac{\alpha}{2}\int_\Omega u^2(x)dx \tag{1.1}$$

subject to

$$-\epsilon\Delta y(x) + \mathbf{a}(x)\cdot\nabla y(x) + r(x)y(x) = f(x)+u(x), \qquad x\in\Omega, \tag{1.2a}$$

$$y(x) = 0, \qquad x\in\partial\Omega_D, \tag{1.2b}$$

$$\epsilon\frac{\partial}{\partial\mathbf{n}}y(x) = g(x), \qquad x\in\partial\Omega_N, \tag{1.2c}$$

where $\partial\Omega_D, \partial\Omega_N$ are boundary segments with $\partial\Omega_D = \partial\Omega\setminus\partial\Omega_N$, $\mathbf{a}, f, g, r, \widehat{y}$ are given functions, $\epsilon, \alpha > 0$ are given scalars, and $\mathbf{n}$ denotes the outward unit normal. Assumptions on these data that ensure the well-posedness of the problem will be given in the next section. The material presented in this paper can be extended to boundary control problems and several other objective functionals. The problem (1.1), (1.2) is an optimization problem in the unknowns y and u, referred to as the state and the control, respectively.

Our domain decomposition method for the solution of (1.1), (1.2) generalizes the Neumann-Neumann domain decomposition method, which is well known for the solution of single PDEs (see, e.g., the books [29, 33, 34]) to the optimization context. Optimization–level Neumann-Neumann domain decomposition methods for elliptic optimal control problems were first introduced in [20, 21] for problems without advection. However, the presence of strong advection can significantly alter the behavior of solution algorithms and typically requires their modification. For domain decomposition methods applied to single advection dominated PDEs a nice overview of this issue is given in [34, Sec. 11.5.1]. The aim of our paper is to tackle this issue for optimal control problems.

The domain decomposition method presented in this paper is formulated at the optimization level. The domain Ω is partitioned into non-overlapping subdomains. Our domain decomposition methods decompose the optimality conditions for (1.1), (1.2). Auxiliary state and so-called adjoints (Lagrange multipliers) are introduced at the subdomain interfaces. The states, adjoints, and controls in the interior of the subdomains are then viewed as implicit functions of the states and adjoints on the interface, defined through the solution of subdomain optimality conditions. To obtain a solution of the original problem (1.1), (1.2), the states and adjoints on the interface have to be chosen such that the implicitly defined states, adjoints, and controls in the interior of the subdomains satisfy certain Robin transmission conditions at the interface boundaries.

These transmission conditions take into account the advection dominated nature of the state equation and are motivated by [3, 4].

The optimization-level domain decomposition described in the previous paragraph leads to a Schur complement formulation for the optimality system. The application of the Schur complement to a given vector of states and adjoints on the interface, requires the parallel solution of subdomain optimal control problems that are essentially copies of (1.1), (1.2) restricted to the subdomains, but with Dirichlet boundary conditions at the subdomain interfaces. The Schur complement is the sum of subdomain Schur complements. Each subdomain Schur complement is shown to be invertible. The application of the inverse of each subdomain Schur complement requires the solution of another subdomain optimal control problem that is also essentially a copy of (1.1), (1.2) restricted to the respective subdomain, but with Robin boundary conditions at the subdomain interfaces. The inverses of the subdomain Schur complements are used to derive preconditioners for the Schur complement.

Section 2 briefly reviews results on the existence, uniqueness and characterization of solutions of (1.1), (1.2). The domain decomposition, interface conditions, subdomain Schur complements and their inverses are discussed in Section 3 using a variational point of view. The corresponding algebraic form, properties of the subdomain Schur complement matrices and some implementation details are presented in Section 4. The performance of the preconditioners on some model problems with distributed control and boundary control are documented in Section 5.

Throughout this paper we use the following notation for norms and inner products. Let $G \subset \Omega \subset \mathbb{R}^d$ or $G \subset \partial\Omega$. We define $\langle f, g\rangle_G = \int_G f(x)g(x)dx$, $\|v\|_{0,G}^2 = \int_G v^2(x)dx$, $|v|_{1,G}^2 = \int_G \nabla v(x) \cdot \nabla v(x)dx$, and $\|v\|_{1,G}^2 = \|v\|_{0,G}^2 + |v|_{1,G}^2$. If $G = \Omega$ we omit G and simply write $\langle f, g\rangle$, etc.

2 The Model Problem

Multiplication of the advection diffusion equation (1.2) by a test function

$$\phi \in Y \stackrel{\text{def}}{=} \left\{\phi \in H^1(\Omega) \,:\, \phi = 0 \text{ on } \partial\Omega_D\right\},$$

integration over Ω, and performing integration by parts leads to the following weak form

$$a(y, \phi) + b(u, \phi) = \langle f, \phi\rangle + \langle g, \phi\rangle_{\partial\Omega_N} \quad \forall \phi \in Y, \tag{2.1}$$

where

$$a(y, \phi) = \int_\Omega \epsilon \nabla y(x) \cdot \nabla \phi(x) + \mathbf{a}(x) \cdot \nabla y(x)\phi(x) + r(x)y(x)\phi(x)dx, \tag{2.2a}$$

$$b(u, \phi) = -\int_\Omega u(x)\phi(x)dx, \tag{2.2b}$$

$$\langle f, \phi\rangle = \int_\Omega f(x)\phi(x)dx, \qquad \langle g, \phi\rangle_{\partial\Omega_N} = \int_{\partial\Omega_N} g(x)\phi(x)dx. \tag{2.2c}$$

We are interested in the solution of the optimal control problem

$$\begin{aligned} &\text{minimize} && \frac{1}{2}\|y-\widehat{y}\|_0^2 + \frac{\alpha}{2}\|u\|_0^2, && \text{(2.3a)}\\ &\text{subject to} && a(y,\phi)+b(u,\phi) = \langle f,\phi\rangle + \langle g,\phi\rangle_{\partial\Omega_N} \quad \forall \phi \in Y, && \text{(2.3b)}\\ & && y\in Y, u\in U, \end{aligned}$$

where the control space is given by $U = L^2(\Omega)$ and the state space Y is as specified above.

We assume that

$$f \in L^2(\Omega), \mathbf{a} \in \left(W^{1,\infty}(\Omega)\right)^2, r \in L^\infty(\Omega), g \in L^2(\partial\Omega_N), \epsilon > 0, \tag{2.4a}$$

$$\partial\Omega_N \subset \{x \in \partial\Omega \,:\, \mathbf{a}(x)\cdot\mathbf{n}(x) \geq 0\} \tag{2.4b}$$

and

$$r(x) - \tfrac{1}{2}\nabla\cdot\mathbf{a}(x) \geq r_0 > 0 \text{ a.e. in } \Omega. \tag{2.4c}$$

If $\partial\Omega_D$ has a nonempty relative interior, then (2.4c) can be replaced by

$$r(x) - \tfrac{1}{2}\nabla\cdot\mathbf{a}(x) \geq r_0 \geq 0 \text{ a.e. in } \Omega. \tag{2.4d}$$

The assumptions (2.4), guarantee that the bilinear form a is continuous on $Y \times Y$ and Y-elliptic (e.g., [28, p. 165], [30, Sec III.1], or [27, Sec. 2.5]). Hence the state equation (2.3b) has a unique solution $y \in Y$ for any given control $u \in U$. The theory in [26, Sec. II.1] guarantees the existence of a unique solution $(y,u) \in Y \times U$ of (2.3).

Theorem 2.1 *If* (2.4) *are satisfied, the optimal control problem* (2.3) *has a unique solution* $(y,u) \in Y \times U$.

The theory in [26, Sec. II.1] also provides necessary and sufficient optimality conditions, which can be best described using the Lagrangian

$$L(y,u,p) = \frac{1}{2}\|y-\widehat{y}\|_0^2 + \frac{\alpha}{2}\|u\|_0^2 + a(y,p) + b(u,p) - \langle f,p\rangle - \langle g,p\rangle_{\partial\Omega_N}. \tag{2.5}$$

The necessary and, for our model problem, sufficient optimality conditions can be obtained by setting the partial Fréchet-derivatives of (2.5) with respect to states $y \in Y$, controls $u \in U$ and adjoints $p \in Y$ equal to zero. This gives the following system consisting of
the adjoint equation

$$a(\psi,p) = -\langle y-\hat{y},\psi\rangle \quad \forall \psi \in Y, \tag{2.6a}$$

the gradient equation

$$b(w,p) + \alpha\langle u,w\rangle = 0 \quad \forall w \in U, \tag{2.6b}$$

and the state equation

$$a(y,\phi) + b(u,\phi) = \langle f,\phi\rangle + \langle g,\phi\rangle_{\partial\Omega_N} \quad \forall \phi \in Y. \tag{2.6c}$$

The gradient equation (2.6b) simply means that

$$p(x) = \alpha u(x) \qquad x \in \Omega \tag{2.7}$$

and (2.6a) is the weak form of

$$-\epsilon \Delta p(x) - \mathbf{a}(x) \cdot \nabla p(x) + (r(x) - \nabla \cdot \mathbf{a}(x)) p(x) = -(y(x) - \hat{y}(x)), \qquad x \in \Omega, \tag{2.8a}$$

$$p(x) = 0, \qquad x \in \partial\Omega_D, \tag{2.8b}$$

$$\epsilon \frac{\partial}{\partial \mathbf{n}} p(x) + \mathbf{a}(x) \cdot \mathbf{n}(x)\, p(x) = 0, \qquad x \in \partial\Omega_N. \tag{2.8c}$$

After finite element discretization, the optimal control problem (2.3) leads to a large-scale linear quadratic optimization problem. It is well known that application of the standard linear finite element method to advection–diffusion equations (1.2) leads to computed solutions with large spurious oscillations, unless the mesh size is sufficiently small relative to the Péclet number (e.g., [28, Sec. 8], [30], or [27]). To allow relatively coarse meshes, we use the streamline upwind/Petrov–Galerkin (SUPG) method [9]. We mention that if the SUPG method, or other stabilized finite element methods are used, the optimality system of the linear quadratic optimization problem corresponding to the discretization of the optimal control problem (2.3) is in general no longer equal to the discretization of the optimality system (2.6). The differences are due to the stabilization term. For a more detailed treatment, we refer to [1, 10]. The papers [1, 10] show that for linear finite elements and suitable choice of the stabilization parameter, these differences are small. In our numerical solution of the problem, we discretize the optimal control problem (2.3) using the SUPG method.

3 Domain Decomposition Schur Complement Formulation of the Example Problem

3.1 Discretization of the Example Problem

We discretize (2.3) using conforming linear finite elements. Let $\{T_l\}$ be a triangulation of Ω and let $\{x_j\}$ be the set of vertices in the triangulation. We divide Ω into nonoverlapping subdomains Ω_i, $i = 1, \ldots, s$, such that each T_l belongs to exactly one $\overline{\Omega}_i$. We define

$$\Gamma_i = \partial\Omega_i \setminus \partial\Omega$$

and

$$\Gamma = \cup_{i=1}^{s} \Gamma_i.$$

The unit outward normal of Ω_i is denoted by $\mathbf{n}_i$. The state y is approximated using piecewise linear functions. We define the finite dimensional state space

$$Y^h = \left\{ \phi_h \in H^1(\Omega) \,:\, \phi_h = 0 \text{ on } \partial\Omega_D,\ \phi_h|_{T_l} \in P^1(T_l) \text{ for all } l \right\}.$$

To formulate our domain decomposition approach we also define

$$\begin{aligned} Y_i^h &= \left\{\phi_h \in H^1(\Omega_i) \,:\, \phi_h = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_D,\ \phi_h|_{T_l} \in P^1(T_l) \text{ for all } T_l \subset \overline{\Omega}_i\right\}, && i = 1,\ldots,s, \\ Y_{i,0}^h &= \left\{\phi_h \in Y_i^h \,:\, \phi_h = 0 \text{ on } \Gamma_i\right\}, && i = 1,\ldots,s, \\ Y_{i,\Gamma_i}^h &= \left\{\phi_h \in Y_i^h \,:\, \phi_h(x_j) = 0, x_j \in \Omega_i \cup (\partial\Omega_i \cap \partial\Omega)\right\}, && i = 1,\ldots,s, \end{aligned} \tag{3.1}$$

and

$$Y_\Gamma^h = \left\{\phi_h \in Y^h \,:\, \phi_h(x_j) = 0, x_j \in \cup_{i=1}^s \Omega_i \cup \partial\Omega\right\}.$$

We identify $(\phi_h)_i \in Y_{i,0}^h$ with a function in Y^h by extending $(\phi_h)_i \in Y_{i,0}^h$ by zero onto Ω. Hence, the state space can be decomposed into $Y^h = Y_\Gamma^h \oplus_{i=1}^s Y_{i,0}^h$.

For our discretization of the control we use piecewise linear functions in Ω. However, our discretization of the control is somewhat nonstandard. A straight forward discretization of the control space by piecewise linear functions would lead to $\left\{u_h \in C^0(\Omega) \,:\, u \in P^1(T_l) \text{ for all } T_l \subset \overline{\Omega}\right\}$. A domain decomposition formulation based on such a discretization would introduce controls associated with the interface Γ, which would have support given by a 'band' of width $O(h)$ around $\partial\Omega_i \cap \partial\Omega_j$, $i \neq j$. Since the evaluation of $u \in L^2(\Omega)$ on $\partial\Omega_i \cap \partial\Omega_j$ does not make sense, we avoid interface controls.

We discretize the control u by a function which is continuous on each Ω_i, $i = 1,\ldots,s$, and linear on each $\Omega_i \cap T_l$. The discretized control is not assumed to be continuous on $\partial\Omega_i \cap \partial\Omega_j$, $i \neq j$. In particular, for each point $x_k \in \partial\Omega_i \cap \partial\Omega_j$, $i \neq j$, there are two discrete controls u_{k_i}, u_{k_j} belonging to subdomains Ω_i and Ω_j, respectively. Because of (2.7) $u_{k_i} - u_{k_j} \to 0$ as $h \to 0$. Since the control space is $L^2(\Omega)$, this is a legitimate discretization. We define the discrete control spaces

$$U_i^h = \left\{u_h \in C^0(\Omega_i) \,:\, u_h \in P^1(T_l) \text{ for all } T_l \subset \overline{\Omega}_i\right\}. \tag{3.2}$$

We identify U_i^h with a subspace of $L^2(\Omega)$ by extending functions $u_i \in U_i^h$ by zero onto Ω. We define

$$U^h = \cup_{i=1}^s U_i^h \subset L^2(\Omega).$$

For advection dominated problems the standard Galerkin method applied to the state equation (2.1) produces strongly oscillatory approximations, unless the mesh size h is chosen sufficiently small relative to $\epsilon/\|\mathbf{a}\|_{0,\infty}$. To obtain approximate solutions of better quality on coarser meshes, various stabilization techniques have been proposed. For an overview see [28, Secs. 8.3.2,8.4] or [30, Sec.3.2]. We use the streamline upwind/Petrov Galerkin (SUPG) method of Hughes and Brooks [9]. The SUPG method computes an approximation $y_h \in Y^h$ of the solution y of the state equation (2.3b) by solving

$$a_h(y_h, \phi_h) + b_h(u_h, \phi_h) = \langle f, \phi_h\rangle_h + \langle g, \phi_h\rangle_{\partial\Omega_N} \quad \forall \phi_h \in Y^h, \tag{3.3}$$

where

$$a_h(y_h, \phi_h) = a(y_h, \phi_h) + \sum_{T_e \in \overline{\Omega}} \tau_e \langle -\epsilon\Delta y_h + \mathbf{a}\cdot\nabla y_h + r y_h, \mathbf{a}\cdot\nabla\phi_h\rangle_{T_e}, \tag{3.4a}$$

$$b_h(u_h, \phi_h) = b(u_h, \phi_h) + \sum_{T_e \in \overline{\Omega}} -\tau_e \langle u_h, \mathbf{a}\cdot\nabla\phi_h\rangle_{T_e}, \tag{3.4b}$$

$$\langle f, \phi_h\rangle_h = \langle f, \phi_h\rangle + \sum_{T_e \in \overline{\Omega}} \tau_e \langle f, \mathbf{a}\cdot\nabla\phi_h\rangle_{T_e}, \tag{3.4c}$$

and $\tau_e > 0$ is a stabilization parameter that is chosen depending on the mesh size and the problem parameters ϵ, $\mathbf{a}$ and r.

Our discretization of the optimal control problem (2.3) is given by

$$\text{minimize} \quad \frac{1}{2}\|y_h - \widehat{y}\|_0^2 + \frac{\alpha}{2}\|u_h\|_0^2, \tag{3.5a}$$

$$\text{subject to} \quad a_h(y_h, \phi_h) + b_h(u_h, \phi_h) = \langle f, \phi_h\rangle_h + \langle g, \phi_h\rangle_{\partial\Omega_N} \quad \forall \phi_h \in Y^h, \tag{3.5b}$$
$$y_h \in Y^h, u_h \in U^h.$$

The necessary and sufficient optimality conditions for (3.5) are given by

$$a_h(\psi_h, p_h) + \langle y_h, \psi_h\rangle = \langle \hat{y}, \psi_h\rangle \qquad \forall \psi_h \in Y^h, \tag{3.6a}$$

$$\alpha\langle u_h, \mu_h\rangle + b_h(\mu_h, p_h) = 0 \qquad \forall \mu_h \in U^h, \tag{3.6b}$$

$$a_h(y_h, \phi_h) + b_h(u_h, \phi_h) = \langle f, \phi_h\rangle_h + \langle g, \phi_h\rangle_{\Gamma_n} \qquad \forall \phi_h \in Y^h. \tag{3.6c}$$

The system (3.6) may also be viewed as a discretization of (2.6). However, as we have discussed already at the end of Section 2, the discretization of the system (2.6) of optimality conditions using SUPG will lead to a slightly different system than (3.6). Everything that follows can be easily applied to the SUPG discretization of the system (2.6) of optimality conditions.

3.2 Domain Decomposition of the Example Problem

To decompose the discrete optimality conditions (3.6), we need local bilinear forms corresponding to the subdomains Ω_i. For advection dominated problems, this requires some care. See, e.g., [34, Sec.11.5.1] for an overview. The straight forward restriction of a defined in (2.2a) to the subdomain Ω_i is given by

$$\widetilde{a}_i(y_h, \phi_h) = \int_{\Omega_i} \epsilon\nabla y_h(x)\cdot\nabla\phi_h(x) + \mathbf{a}(x)\cdot\nabla y_h(x)\phi_h(x) + r(x)y_h(x)\phi_h(x)dx. \tag{3.7}$$

Integration by parts and application of the chain rule to $\nabla\cdot(\mathbf{a}(x)\phi_h(x))$ show that

$$\begin{aligned}\widetilde{a}_i(y_h, \phi_h) \;=\; & \int_{\Omega_i} \epsilon\nabla y_h(x)\cdot\nabla\phi_h(x) + \tfrac{1}{2}\mathbf{a}(x)\cdot\nabla y_h(x)\phi_h(x) \\ & \quad -\tfrac{1}{2}\mathbf{a}(x)\cdot\nabla\phi_h(x)y_h(x) + (r(x) - \tfrac{1}{2}\nabla\cdot\mathbf{a}(x))y_h(x)\phi_h(x)dx \\ & +\tfrac{1}{2}\int_{\partial\Omega_i\cap\partial\Omega_N} \mathbf{a}(x)\cdot\mathbf{n}_i\, y_h(x)\phi_h(x)dx + \tfrac{1}{2}\int_{\partial\Omega_i\setminus\partial\Omega} \mathbf{a}(x)\cdot\mathbf{n}_i\, y_h(x)\phi_h(x)dx\end{aligned}$$

for all $y_h, \phi_h \in Y_i^h$. Because of the last boundary integral, the assumptions (2.4) no longer guarantee that a_i is Y_i^h–elliptic. Hence, we follow [4] and use the local bilinear form

$$\begin{aligned} a_i(y_h, \phi_h) \;=\; & \widetilde{a}_i(y_h, \phi_h) - \tfrac{1}{2}\int_{\partial\Omega_i\setminus\partial\Omega} \mathbf{a}(x)\cdot\mathbf{n}_i\, y_h(x)\phi_h(x)dx \\ =\; & \int_{\Omega_i} \epsilon\nabla y_h(x)\cdot\nabla\phi_h(x) + \tfrac{1}{2}\mathbf{a}(x)\cdot\nabla y_h(x)\phi_h(x) \\ & \quad -\tfrac{1}{2}\mathbf{a}(x)\cdot\nabla\phi_h(x)y_h(x) + (r(x) - \tfrac{1}{2}\nabla\cdot\mathbf{a}(x))y_h(x)\phi_h(x)dx \\ & +\tfrac{1}{2}\int_{\partial\Omega_i\cap\partial\Omega_N} \mathbf{a}(x)\cdot\mathbf{n}_i\, y_h(x)\phi_h(x)dx. \end{aligned} \tag{3.8}$$

Note that

$$\sum_{i=1}^{s} \tfrac{1}{2} \int_{\partial\Omega_i \setminus \partial\Omega} \mathbf{a}(x) \cdot \mathbf{n}_i \, y_h(x)\phi_h(x)dx = \sum_{i=1}^{s} \sum_{j \neq i} \tfrac{1}{2} \int_{\partial\Omega_i \cap \partial\Omega_j} \mathbf{a}(x) \cdot \mathbf{n}_i \, y_h(x)\phi_h(x)dx = 0$$

since each boundary integral $\int_{\partial\Omega_i \cap \partial\Omega_j}$ appears twice in the summation, once with integrand $\mathbf{a}(x) \cdot \mathbf{n}_i \, y_h(x)\phi_h(x)$, the other time with integrand $\mathbf{a}(x) \cdot \mathbf{n}_j \, y_h(x)\phi_h(x) = -\mathbf{a}(x) \cdot \mathbf{n}_i \, y_h(x)\phi_h(x)$. Hence

$$\sum_{i=1}^{s} a_i(y_h, \phi_h) = \sum_{i=1}^{s} \widetilde{a}_i(y_h, \phi_h) = a(y_h, \phi_h) \quad \forall y_h, \phi_h \in Y_i^h,$$

i.e., the global problem is not altered.

Accounting for the SUPG terms, we define

$$\begin{aligned} a_{i,h}(y_h, \phi_h) &= \widetilde{a}_i(y_h, \psi_h) - \tfrac{1}{2} \int_{\partial\Omega_i \setminus \partial\Omega_N} \mathbf{a}(x) \cdot \mathbf{n}_i \, y_h(x)\phi_h(x)dx \\ &\quad + \sum_{T_e \in \overline{\Omega_i}} \tau_e \langle -\epsilon \Delta y_h + \mathbf{a} \cdot \nabla y_h + r y_h, \mathbf{a} \cdot \nabla \phi_h \rangle_{T_e}, \qquad (3.9a) \\ b_{i,h}(u_h, \phi_h) &= -\langle u_h, \phi_h \rangle_{\Omega_i} + \sum_{T_e \in \overline{\Omega_i}} -\tau_e \langle u_h, \mathbf{a} \cdot \nabla \phi_h \rangle_{T_e}, \qquad (3.9b) \\ \langle f, \phi_h \rangle_{\Omega_i, h} &= \langle f, \phi_h \rangle_{\Omega_i} + \sum_{T_e \in \overline{\Omega_i}} \tau_e \langle f, \mathbf{a} \cdot \nabla \phi_h \rangle_{T_e}, \qquad (3.9c) \end{aligned}$$

Now, we decompose the optimality system (3.6) by introducing artificial state and adjoint variables $y_\Gamma, p_\Gamma \in Y_\Gamma^h$ on the subdomain interfaces. Given $y_\Gamma, p_\Gamma \in Y_\Gamma^h$ we consider

$$\begin{aligned} a_{i,h}(\psi_i, p_i) + \langle y_i, \psi_i \rangle_{\Omega_i} &= \langle \hat{y}, \psi_i \rangle_{\Omega_i} && \forall \psi_i \in Y_{i,0}^h, && (3.10a) \\ b_{i,h}(\mu_i, p_i) + \langle u_i, \mu_i \rangle_{\Omega_i} &= 0 && \forall \mu_i \in U_i^h, && (3.10b) \\ a_{i,h}(y_i, \phi_i) + b_{i,h}(u_i, \phi_i) &= \langle f, \phi_i \rangle_{\Omega_i, h} + \langle g, \phi_i \rangle_{\partial\Omega_i \cap \partial\Omega_N} && \forall \phi_i \in Y_{i,0}^h, && (3.10c) \\ y_i = y_\Gamma, \quad p_i &= p_\Gamma && \text{on } \Gamma_i, && (3.10d) \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^{s} a_{i,h}(y_i, \mathcal{R}_h^a(v_\Gamma, q_\Gamma)) + b_{i,h}(u_i, \mathcal{R}_h^a(v_\Gamma, q_\Gamma)) + a_{i,h}(\mathcal{R}_h^s(v_\Gamma, q_\Gamma), p_i) + \langle y_i, \mathcal{R}_h^s(v_\Gamma, q_\Gamma) \rangle_{\Omega_i} \\ = \; &\sum_{i=1}^{s} \langle f, \mathcal{R}_h^a(v_\Gamma, q_\Gamma) \rangle_{\Omega_i, h} + \langle g, \mathcal{R}_h^a(v_\Gamma, q_\Gamma) \rangle_{\partial\Omega_i \cap \partial\Omega_N} + \langle \hat{y}_i, \mathcal{R}_h^s(v_\Gamma, q_\Gamma) \rangle_{\Omega_i} \qquad (3.11) \end{aligned}$$

for all $v_\Gamma, q_\Gamma \in Y_\Gamma^h$, where

$$\mathcal{R}_h^s, \mathcal{R}_h^a : Y_\Gamma^h \times Y_\Gamma^h \to Y^h \qquad (3.12a)$$

are continuous linear extension operators with

$$\mathcal{R}_h^s(v_\Gamma, q_\Gamma)(x) = v_\Gamma(x), \quad \mathcal{R}_h^a(v_\Gamma, q_\Gamma)(x) = q_\Gamma(x), \quad \text{for all } x \in \Gamma \text{ and for all } v_\Gamma, q_\Gamma \in Y_\Gamma^h. \qquad (3.12b)$$

Theorem 3.1 *If* $(y, u, p) \in Y^h \times U^h \times Y^h$ *solves* (3.6)*, then* $y_i = y|_{\Omega_i}$, $u_i = u|_{\Omega_i}$, $p_i = p|_{\Omega_i}$, $i = 1, \ldots, s$, *solve* (3.10), (3.11).

If $(y_\Gamma, p_\Gamma) \in Y_\Gamma^h \times Y_\Gamma^h$ *is such that the solution* $(y_i, u_i, p_i) \in Y_i^h \times U_i^h \times Y_i^h$*, of* (3.10), $i = 1, \ldots, s$, *satisfies the interface conditions* (3.11)*, then* $(y, u, p) \in Y^h \times U^h \times Y^h$ *given by* $y|_{\Omega_i} = y_i$, $u|_{\Omega_i} = u_i$, $p|_{\Omega_i} = p_i$, $i = 1, \ldots, s$, *solves* (3.6).

The proof of this result is analogous to the proof of [29, Lemma 1.2.1] and is omitted.

We will view the solution of (3.10) as an affine linear function of (y_Γ, p_Γ) and then consider (3.11) as a linear equation in (y_Γ, p_Γ). We will describe this process using the variational formulation in Subsection 3.3 and we will describe the algebraic version in Section 4. The latter is used computationally. Section 4 can be read without knowledge of the material in the remainder of this section. The main purpose of the remainder of this section is to connect the subproblems that need to be solved to the original optimal control problem (1.1).

We close this subsection with an interpretation of (3.10) and (3.11).

Remark 3.2 *i. The systems* (3.10), $i = 1, \ldots, s$, *can be interpreted as the finite element discretization of*

$$-\epsilon\Delta y_i(x) + \mathbf{a}(x)\cdot\nabla y_i(x) + r(x)y_i(x) = f(x) + u_i(x) \qquad \text{in } \Omega_i, \tag{3.13a}$$
$$y_i(x) = 0 \qquad \text{on } \partial\Omega_i \cap \partial\Omega_D, \tag{3.13b}$$
$$\epsilon\frac{\partial}{\partial\mathbf{n}}y_i(x) = g(x), \qquad \text{on } \partial\Omega_i \cap \partial\Omega_N, \tag{3.13c}$$
$$y_i(x) = y_\Gamma(x) \qquad \text{on } \Gamma_i, \tag{3.13d}$$
$$-\epsilon\Delta p_i(x) - \mathbf{a}(x)\cdot\nabla p_i(x) + (r(x) - \nabla\cdot\mathbf{a}(x))p_i(x) = -(y_i(x) - \hat{y}(x)) \qquad \text{in } \Omega_i, \tag{3.13e}$$
$$p_i(x) = 0, \qquad \text{on } \partial\Omega_i \cap \partial\Omega_D, \tag{3.13f}$$
$$\epsilon\frac{\partial}{\partial\mathbf{n}}p_i(x) + \mathbf{a}(x)\cdot\mathbf{n}(x)\, p_i(x) = 0, \qquad \text{on } \partial\Omega_i \cap \partial\Omega_N, \tag{3.13g}$$
$$p_i(x) = p_\Gamma(x) \qquad \text{on } \Gamma_i, \tag{3.13h}$$
$$\alpha u_i(x) - p_i(x) = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_i. \tag{3.13i}$$

Applying the arguments in [21] to the advection diffusion case, the system (3.13) *may be viewed as the necessary and sufficient optimality conditions for*

$$\text{minimize } \tfrac{1}{2}\int_{\Omega_i}(y_i(x) - \hat{y}(x))^2 dx + \frac{\alpha}{2}\int_{\Omega_i} u_i^2(x)dx + \int_{\Gamma_i}\left(\epsilon\frac{\partial}{\partial\mathbf{n}_i} - \tfrac{1}{2}\mathbf{a}(x)\mathbf{n}_i\right) y_i(x)p_\Gamma(x)dx, \tag{3.14a}$$

subject to

$$-\epsilon\Delta y_i(x) + \mathbf{a}(x)\cdot\nabla y_i(x) + r(x)y_i(x) = f(x) + u_i(x) \qquad \text{in } \Omega_i, \tag{3.14b}$$
$$y_i(x) = 0 \qquad \text{on } \partial\Omega_i \cap \partial\Omega_D, \tag{3.14c}$$
$$\epsilon\frac{\partial}{\partial\mathbf{n}}y_i(x) = g(x), \qquad \text{on } \partial\Omega_i \cap \partial\Omega_N, \tag{3.14d}$$
$$y_i(x) = y_\Gamma(x) \qquad \text{on } \Gamma_i, \tag{3.14e}$$

ii. The interface condition (3.11) *can be interpreted as*

$$\begin{array}{rcll} \left(\epsilon\frac{\partial}{\partial \mathbf{n}_i} - \frac{1}{2}\mathbf{a}(x)\mathbf{n}_i\right) y_i(x) & = & -\left(\epsilon\frac{\partial}{\partial \mathbf{n}_j} - \frac{1}{2}\mathbf{a}(x)\mathbf{n}_j\right) y_j(x) & x \in \partial\Omega_i \cap \partial\Omega_j, \\ \left(\epsilon\frac{\partial}{\partial \mathbf{n}_i} + \frac{1}{2}\mathbf{a}(x)\mathbf{n}_i\right) p_i(x) & = & -\left(\epsilon\frac{\partial}{\partial \mathbf{n}_j} + \frac{1}{2}\mathbf{a}(x)\mathbf{n}_j\right) p_j(x) & x \in \partial\Omega_i \cap \partial\Omega_j, \end{array} \tag{3.15}$$

for $i, j = 1, \ldots, s$, $i \neq j$.

Remark 3.3 *We briefly comment on the subproblems that would arise if we had used the unmodified local bilinear forms* $a_{i,h}(y_h, \phi_h) = \widetilde{a}_i(y_h, \psi_h)$ *instead of* (3.9).

i. If $a_{i,h}(y_h, \phi_h) = \widetilde{a}_i(y_h, \psi_h)$, *the systems* (3.10), $i = 1, \ldots, s$, *can still be interpreted as the finite element discretization of* (3.13) *which, in turn, can be viewed as the optimality conditions for* (3.14).

ii. If $a_{i,h}(y_h, \phi_h) = \widetilde{a}_i(y_h, \psi_h)$, *the interface condition* (3.11) *can be interpreted as*

$$\begin{array}{rcll} \epsilon\frac{\partial}{\partial \mathbf{n}_i} y_i(x) & = & -\epsilon\frac{\partial}{\partial \mathbf{n}_j} y_j(x) & x \in \partial\Omega_i \cap \partial\Omega_j, \\ \left(\epsilon\frac{\partial}{\partial \mathbf{n}_i} + \mathbf{a}(x)\mathbf{n}_i\right) p_i(x) & = & -\left(\epsilon\frac{\partial}{\partial \mathbf{n}_j} + \mathbf{a}(x)\mathbf{n}_j\right) p_j(x) & x \in \partial\Omega_i \cap \partial\Omega_j, \end{array} \tag{3.16}$$

for $i, j = 1, \ldots, s$, $i \neq j$.

3.3 Schur Complement Formulation

As we have stated earlier, we will view the solution of (3.10) as an affine linear function of (y_Γ, p_Γ) and then consider (3.11) as a linear equation in (y_Γ, p_Γ). The variational formulation of this process is studied here. It complements Section 4, but is not required for the reading of Section 4.

First, we specify the extension operators $\mathcal{R}_h^s$ and $\mathcal{R}_h^a$ that we use in (3.11). They are generalizations of the so-called harmonic extensions used in the PDE case to the optimal control setting. For $i = 1, \ldots, s$, we define the linear operators

$$\mathcal{H}_i^h : Y_\Gamma^h \times Y_\Gamma^h \to Y_i^h \times U_i^h \times Y_i^h \tag{3.17a}$$

with

$$\mathcal{H}_i^h(y_\Gamma, p_\Gamma) = \begin{pmatrix} (\mathcal{H}_i^h)^y(y_\Gamma, p_\Gamma) \\ (\mathcal{H}_i^h)^u(y_\Gamma, p_\Gamma) \\ (\mathcal{H}_i^h)^p(y_\Gamma, p_\Gamma) \end{pmatrix} = \begin{pmatrix} y_i^0 \\ u_i^0 \\ p_i^0 \end{pmatrix}, \tag{3.17b}$$

where (y_i^0, u_i^0, p_i^0) is the solution of (3.10) with $f = 0$, $g = 0$ and $\hat{y} = 0$. We consider (3.11) with $\mathcal{R}_h^s = (\mathcal{H}_i^h)^y$ and $\mathcal{R}_h^a = (\mathcal{H}_i^h)^p$.

With this choice of the extension operators, the left hand side in (3.11) defines a continuous bilinear form on $(Y_\Gamma^h)^2 \times (Y_\Gamma^h)^2$, or, equivalently, a bounded linear operator $\mathcal{S} : (Y_\Gamma^h)^2 \to ((Y_\Gamma^h)^*)^2$, where $(Y_\Gamma^h)^*$ denotes the dual of Y_Γ^h. This operator can be expressed as the sum of $\mathcal{S}_i^h$'s given as follows. Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denote the duality pairing between $(Y_\Gamma^h)^2$ and $((Y_\Gamma^h)^*)^2$. We define the linear subdomain Schur complement operator

$$\mathcal{S}_i^h : (Y_\Gamma^h)^2 \to ((Y_\Gamma^h)^*)^2, \tag{3.18a}$$

$i = 1, \ldots, s$, with

$$\begin{aligned} &\langle\!\langle \mathcal{S}_i^h(y_\Gamma, p_\Gamma), (v_\Gamma, q_\Gamma) \rangle\!\rangle \\ &\quad = \; a_{i,h}((\mathcal{H}_i^h)^y(y_\Gamma, p_\Gamma), (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)) + b_{i,h}((\mathcal{H}_i^h)^u(y_\Gamma, p_\Gamma), (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)) \\ &\qquad + a_{i,h}((\mathcal{H}_i^h)^y(v_\Gamma, q_\Gamma), (\mathcal{H}_i^h)^p(y_\Gamma, p_\Gamma)) + \langle (\mathcal{H}_i^h)^y(y_\Gamma, p_\Gamma), (\mathcal{H}_i^h)^y(v_\Gamma, q_\Gamma) \rangle_{\Omega_i}. \end{aligned} \tag{3.18b}$$

Moreover, we define $r_i \in ((Y_\Gamma^h)^*)^2$, $i = 1, \ldots, s$, as

$$\begin{aligned}\langle\!\langle r_i, (v_\Gamma, q_\Gamma)\rangle\!\rangle &= \langle f, (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)\rangle_{\Omega_i,h} + \langle g, (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)\rangle_{\partial\Omega_i\cap\partial\Omega_N} + \langle \hat{y}_i, (\mathcal{H}_i^h)^y(v_\Gamma, q_\Gamma)\rangle_{\Omega_i} \\ &\quad -a_{i,h}(y_i, (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)) - b_{i,h}(u_i, (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)) \\ &\quad -a_{i,h}((\mathcal{H}_i^h)^y(v_\Gamma, q_\Gamma), p_i) - \langle y_i, (\mathcal{H}_i^h)^y(v_\Gamma, q_\Gamma)\rangle_{\Omega_i}, \end{aligned} \tag{3.19}$$

where (y_i, u_i, p_i) is the solution of (3.10) with $y_\Gamma = 0$ and $p_\Gamma = 0$.

Theorem 3.1 with $\mathcal{R}_h^s(v_\Gamma, q_\Gamma)|_{\Omega_i} = (\mathcal{H}_i^h)^y(v_\Gamma, q_\Gamma)$ and $\mathcal{R}_h^a(v_\Gamma, q_\Gamma)|_{\Omega_i} = (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)$ implies that the system (3.6) of optimality conditions is equivalent to the Schur complement system

$$\sum_{i=1}^s \mathcal{S}_i^h(y_\Gamma, p_\Gamma) = \sum_{i=1}^s r_i \quad \text{in } ((Y_\Gamma^h)^*)^2. \tag{3.20}$$

The next result establishes the invertibility of the subdomain Schur complement operator $\mathcal{S}_i$.

Theorem 3.4 *Let $r_i = (r_i^y, r_i^p) \in ((Y_{i,\Gamma_i}^h)^*)^2$.*

i. If (2.4a)–(2.4c) *hold and if the stabilization parameter τ_e is sufficiently small, then the unique solution $(y_\Gamma, p_\Gamma) \in (Y_{i,\Gamma_i}^h)^2$ of*

$$\mathcal{S}_i(y_\Gamma, p_\Gamma) = r_i \tag{3.21}$$

is given by

$$y_\Gamma = y_i|_{\Gamma_i}, \quad p_\Gamma = p_i|_{\Gamma_i},$$

where $(y_i, u_i, p_i) \in Y_i^h \times U_i^h \times Y_i^h$ is the unique solution of

$$a_{i,h}(\psi, p_i) + \langle y_i, \psi\rangle_{\Omega_i} = \langle r_i^y, \psi\rangle_{\Gamma_i} \qquad \forall \psi \in Y_i^h, \tag{3.22a}$$

$$b_{i,h}(\mu, p_i) + \alpha\langle u_i, \mu\rangle_{\Omega_i} = 0 \qquad \forall \mu \in U_i^h, \tag{3.22b}$$

$$a_{i,h}(y_i, \psi) + b_{i,h}(u_i, \psi) = \langle r_i^p, \psi\rangle_{\Gamma_i} \qquad \forall \psi \in Y_i^h. \tag{3.22c}$$

ii. If the relative interior of $\partial\Omega_i \cap \partial\Omega_D$ is nonempty, then the assumption (2.4c) *in part i. can be replaced by* (2.4d).

Proof: By definition (3.18) of $\mathcal{S}_i$, the equality (3.21) can be written as

$$\begin{aligned} & a_{i,h}((\mathcal{H}_i^h)^y(y_\Gamma, p_\Gamma), (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)) + b_{i,h}((\mathcal{H}_i^h)^u(y_\Gamma, p_\Gamma), (\mathcal{H}_i^h)^p(v_\Gamma, q_\Gamma)) \\ & +a_{i,h}((\mathcal{H}_i^h)^y(v_\Gamma, q_\Gamma), (\mathcal{H}_i^h)^p(y_\Gamma, p_\Gamma)) + \langle (\mathcal{H}_i^h)^y(y_\Gamma, p_\Gamma), (\mathcal{H}_i^h)^y(v_\Gamma, q_\Gamma)\rangle_{\Omega_i} \\ = \; & \langle r_i^y, v_\Gamma\rangle_{\Gamma_i} + \langle r_i^p, q_\Gamma\rangle_{\Gamma_i} \end{aligned} \tag{3.23}$$

for all $v_\Gamma, q_\Gamma \in Y_{i,\Gamma_i}^h$. Using the definition (3.17) of $\mathcal{H}_i(y_\Gamma, p_\Gamma)$ together with (3.23), we see that (3.21) is

equivalent to

$$
\begin{aligned}
&a_{i,h}((\mathcal{H}_i^h)^y(y_\Gamma,p_\Gamma),(\mathcal{H}_i^h)^p(v_\Gamma,q_\Gamma))\\
&+b_{i,h}((\mathcal{H}_i^h)^u(y_\Gamma,p_\Gamma),(\mathcal{H}_i^h)^p(v_\Gamma,q_\Gamma))\\
&+a_{i,h}((\mathcal{H}_i^h)^y(v_\Gamma,q_\Gamma),(\mathcal{H}_i^h)^p(y_\Gamma,p_\Gamma))\\
&+\langle(\mathcal{H}_i^h)^y(y_\Gamma,p_\Gamma),(\mathcal{H}_i^h)^y(v_\Gamma,q_\Gamma)\rangle_{\Omega_i}=\langle r_i^y,v_\Gamma\rangle_{\Gamma_i}+\langle r_i^p,q_\Gamma\rangle_{\Gamma_i} && \forall v_\Gamma,q_\Gamma\in Y_{i,\Gamma_i}^h, && (3.24a)\\
&a_{i,h}(\psi^0,p_i)+\langle y_i,\psi^0\rangle_{\Omega_i}=0 && \forall\psi^0\in Y_{i,0}^h, && (3.24b)\\
&b_{i,h}(\mu,p_i)+\alpha\langle u_i,\mu\rangle_{\Omega_i}=0 && \forall\mu\in U_i^h, && (3.24c)\\
&a_{i,h}(y_i,\phi^0)+b_{i,h}(u_i,\phi^0)=0 && \forall\phi^0\in Y_{i,0}^h, && (3.24d)\\
&y_i=y_\Gamma,\quad p_i=p_\Gamma && \text{on }\Gamma, && (3.24e)
\end{aligned}
$$

If we set $\psi=\psi^0+(\mathcal{H}_i^h)^y(y_\Gamma,p_\Gamma)\in Y_i^h$ and $\phi=\phi^0+(\mathcal{H}_i^h)^p(y_\Gamma,p_\Gamma)\in Y_i^h$, then (3.24) is equivalent to

$$
\begin{aligned}
&a_{i,h}(\psi,p_i)+\langle y_i,\psi\rangle_{\Omega_i}=\langle r_i^y,\psi\rangle_{\Gamma_i}, && \forall\psi\in Y_i^h, && (3.25a)\\
&b_{i,h}(\mu,p_i)+\alpha\langle u_i,\mu\rangle_{\Omega_i}=0 && \forall\mu\in U_i^h, && (3.25b)\\
&a_{i,h}(y_i,\phi)+b_{i,h}(u_i,\phi)=\langle r_i^p,\phi\rangle_{\Gamma_i} && \forall\phi\in Y_i^h, && (3.25c)\\
&y_i=y_\Gamma,\quad p_i=p_\Gamma && \text{on }\Gamma, && (3.25d)
\end{aligned}
$$

The assertion follows if we prove that (3.22) has a unique solution $(y_i,u_i,p_i)\in Y_i^h\times U_i^h\times Y_i^h$. Let $(y_i^1,u_i^1,p_i^1),(y_i^2,u_i^2,p_i^2)\in Y_i^h\times U_i^h\times Y_i^h$ be solutions of (3.22). Then $(e_i^y,e_i^u,e_i^p)=(y_i^1-y_i^2,u_i^1-u_i^2,p_i^1-p_i^2)\in Y_i^h\times U_i^h\times Y_i^h$ satisfies

$$
\begin{aligned}
&a_{i,h}(\psi,e_i^p)+\langle e_i^y,\psi\rangle_{\Omega_i}=0 && \forall\psi\in Y_i^h, && (3.26a)\\
&b_{i,h}(\mu,e_i^p)+\alpha\langle e_i^u,\mu\rangle_{\Omega_i}=0 && \forall\mu\in U_i^h, && (3.26b)\\
&a_{i,h}(e_i^y,\phi)+b_{i,h}(e_i^u,\phi)=0 && \forall\phi\in Y_i^h. && (3.26c)
\end{aligned}
$$

If we set $\psi=e_i^y$, $\mu=e_i^u$, and $\phi=-e_i^p$ in (3.26) and add the resulting equations, we obtain

$$0=\|e_i^y\|_{0,\Omega_i}^2+\alpha\|e_i^u\|_{0,\Omega_i}^2.$$

Hence $e_i^y=0$ and $e_i^u=0$. Now, consider (3.26a) with $\psi=e_i^p$. Using the definitions (3.8) and (3.9) and the assumptions (2.4a), (2.4b), (2.4d) we have

$$
\begin{aligned}
0=a_{i,h}(e_i^p,e_i^p) &= \int_{\Omega_i}\epsilon\nabla e_i^p(x)\cdot\nabla e_i^p(x)+(r(x)-\tfrac{1}{2}\nabla\cdot\mathbf{a}(x))(e_i^p(x))^2dx\\
&\quad+\tfrac{1}{2}\int_{\partial\Omega_i\cap\partial\Omega_N}\mathbf{a}(x)\cdot\mathbf{n}_i\,(e_i^p(x))^2dx,\\
&\quad+\sum_{T_e\in\overline{\Omega_i}}\tau_e\langle-\epsilon\Delta e_i^p+\mathbf{a}\cdot\nabla e_i^p+re_i^p,\mathbf{a}\cdot\nabla e_i^p\rangle_{T_e}\\
&\geq \epsilon|\nabla e_i^p|_{1,\Omega_i}+r_0\|e_i^p\|_{0,\Omega_i}\\
&\quad+\sum_{T_e\in\overline{\Omega_i}}\tau_e\langle-\epsilon\Delta e_i^p+\mathbf{a}\cdot\nabla e_i^p+re_i^p,\mathbf{a}\cdot\nabla e_i^p\rangle_{T_e}.
\end{aligned}
$$

(Note that the modification of the local bilinear from (3.8) was used to derive the previous inequality.) Standard SUPG estimates (cf. [25, p. 378] or [30, L. 3.28,p. 231]) show that

$$0 = a_{i,h}(e_i^p, e_i^p) \geq \frac{\epsilon}{2}|\nabla e_i^p|_{1,\Omega_i} + \frac{r_0}{2}\|e_i^p\|_{0,\Omega_i} + \frac{1}{2}\sum_{T_e \in \overline{\Omega_i}} \tau_e \|\mathbf{a}\cdot\nabla e_i^p\|_{0,T_e},$$

for sufficiently small τ_e. This implies $e_i^p = 0$.

Part ii. can be proven analogously. □

Remark 3.5 *i. Equations* (3.22) *can be interpreted as the weak form of*

$$-\epsilon\Delta y_i(x) + \mathbf{a}(x)\cdot\nabla y_i(x) + r(x)y_i(x) = u_i(x) \qquad \text{in } \Omega_i, \tag{3.27a}$$

$$y_i(x) = 0 \qquad \text{on } \partial\Omega_i \cap \partial\Omega_D, \tag{3.27b}$$

$$\epsilon\frac{\partial}{\partial \mathbf{n}_i} y_i(x) = 0 \qquad \text{on } \partial\Omega_i \cap \partial\Omega_N, \tag{3.27c}$$

$$\left(\epsilon\frac{\partial}{\partial \mathbf{n}_i} - \tfrac{1}{2}\mathbf{a}(x)\cdot\mathbf{n}_i\right) y_i(x) = r_i^y(x) \qquad \text{on } \Gamma_i, \tag{3.27d}$$

$$-\epsilon\Delta p_i(x) - \mathbf{a}(x)\cdot\nabla p_i(x) + (r(x) - \nabla\cdot\mathbf{a}(x))p_i(x) = -y_i(x) \qquad \text{in } \Omega_i, \tag{3.27e}$$

$$p_i(x) = 0, \qquad \text{on } \partial\Omega_i \cap \partial\Omega_D, \tag{3.27f}$$

$$\epsilon\frac{\partial}{\partial \mathbf{n}_i} p_i(x) + \mathbf{a}(x)\cdot\mathbf{n}(x)\, p_i(x) = 0, \qquad \text{on } \partial\Omega_i \cap \partial\Omega_N, \tag{3.27g}$$

$$\left(\epsilon\frac{\partial}{\partial \mathbf{n}_i} + \tfrac{1}{2}\mathbf{a}(x)\cdot\mathbf{n}_i\right) p_i(x) = r_i^p(x) \qquad \text{on } \Gamma_i, \tag{3.27h}$$

$$\alpha u_i(x) - p_i(x) = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_i \tag{3.27i}$$

The terms $\frac{1}{2}\mathbf{a}(x)\cdot\mathbf{n}_i$ *in (3.27d,h) arise because of the modification* (3.8) *in the local bilinear form* $a_{i,h}$.

ii. The system (3.27) *may be viewed as the necessary and sufficient optimality conditions for*

$$\text{minimize } \tfrac{1}{2}\int_{\Omega_i} y_i^2(x)dx + \frac{\alpha}{2}\int_{\Omega_i} u_i^2(x)dx - \int_{\Gamma_i} y_i(x) r_i^p(x)dx, \tag{3.28a}$$

subject to

$$-\epsilon\Delta y_i(x) + \mathbf{a}(x)\cdot\nabla y_i(x) + r(x)y_i(x) = u_i(x) \qquad \text{in } \Omega_i, \tag{3.28b}$$

$$y_i(x) = 0 \qquad \text{on } \partial\Omega_i \cap \partial\Omega_D, \tag{3.28c}$$

$$\epsilon\frac{\partial}{\partial \mathbf{n}} y_i(x) = 0, \qquad \text{on } \partial\Omega_i \cap \partial\Omega_N, \tag{3.28d}$$

$$\left(\epsilon\frac{\partial}{\partial \mathbf{n}_i} - \tfrac{1}{2}\mathbf{a}(x)\cdot\mathbf{n}_i\right) y_i(x) = r_i^y(x) \qquad \text{on } \Gamma_i, \tag{3.28e}$$

Remark 3.6 *If the unmodified local bilinear forms $a_{i,h}(y_h, \phi_h) = \widetilde{a}_i(y_h, \psi_h)$ were used instead of* (3.9)*, the invertibility of $\mathcal{S}_i$ can no longer be guaranteed in general.*

However, if $a_{i,h}(y_h, \phi_h) = \widetilde{a}_i(y_h, \psi_h)$, and if $\mathcal{S}_i$ is invertible, then the application of its inverse corresponds to the solution of (3.29) *with (3.29d) and (3.29h) replaced by*

$$\epsilon \frac{\partial}{\partial \mathbf{n}_i} y_i(x) = r_i^y(x) \qquad \text{on } \Gamma_i, \tag{3.29a}$$

$$\left(\epsilon \frac{\partial}{\partial \mathbf{n}_i} + \mathbf{a}(x) \cdot \mathbf{n}_i\right) p_i(x) = r_i^p(x) \qquad \text{on } \Gamma_i, \tag{3.29b}$$

respectively.

4 Algebraic Formulation

The discretization of the optimal control problem (3.5) using piecewise linear finite elements with SUPG stabilization leads to a large-scale linear quadratic problem of the form

$$\text{minimize } \frac{1}{2}\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{c}^T\mathbf{y} + \frac{\alpha}{2}\mathbf{u}^T\mathbf{R}\mathbf{u}, \tag{4.1a}$$

$$\text{subject to } \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} = \mathbf{b}. \tag{4.1b}$$

For the model problem, the matrices $\mathbf{Q} \in \mathbb{R}^{m\times m}, \mathbf{R} \in \mathbb{R}^{n\times n}$ are mass matrices and are symmetric positive definite. The stiffness matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is non-symmetric, but, under the assumptions (2.4) and with sufficiently small stabilization parameter τ_e (cf. [25, p. 378] or [30, L. 3.28,p. 231]), the matrix obeys $\mathbf{y}^T\mathbf{A}\mathbf{y} > 0$ for all $\mathbf{y} \neq 0$. In particular under these conditions $\mathbf{A}$ is invertible. The necessary and sufficient optimality conditions for (4.1) are given by

$$\begin{pmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{A}^T \\ \mathbf{0} & \alpha\mathbf{R} & \mathbf{B}^T \\ \mathbf{A} & \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} -\mathbf{c} \\ \mathbf{0} \\ \mathbf{b} \end{pmatrix}. \tag{4.2}$$

The system matrix in (4.2) is symmetric indefinite and has $m + n$ positive eigenvalues and m negative eigenvalues [14].

4.1 Domain Decomposition Schur Complement Formulation

We can use the decomposition of Ω to decompose the matrices $\mathbf{A}$, etc. Our notation follows the commonly used in the domain decomposition literature, see, e.g., [29, Sec. 2.3] [33, Sec. 4], [34, Sec. 1.2]. Let $\mathbb{I}_i^y$, $i = 1, \ldots, s$, be the restriction operator which maps from the vector of coefficient unknowns on the interface boundary, $\mathbf{y}_\Gamma$, to only those associated with the boundary of Ω_i, and let

$$\mathbb{I}_i = \begin{pmatrix} \mathbb{I}_i^y & \\ & \mathbb{I}_i^p \end{pmatrix}, \quad \mathbb{I}_i^p = \mathbb{I}_i^y. \tag{4.3}$$

After a suitable reordering of rows and columns, the stiffness matrix can be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{II}^1 & & & \mathbf{A}_{I\Gamma}^1 \mathbb{I}_1^y \\ & \ddots & & \vdots \\ & & \mathbf{A}_{II}^s & \mathbf{A}_{I\Gamma}^s \mathbb{I}_s^y \\ (\mathbb{I}_1^y)^T \mathbf{A}_{\Gamma I}^1 & \cdots & (\mathbb{I}_s^y)^T \mathbf{A}_{\Gamma I}^s & \mathbf{A}_{\Gamma\Gamma} \end{pmatrix}. \tag{4.4}$$

$\mathbf{A}_{\Gamma\Gamma} = \sum_{i=1}^s (\mathbb{I}_i^y)^T \mathbf{A}_{\Gamma\Gamma}^i \mathbb{I}_i^y$. Similar decompositions can be introduced for $\mathbf{Q}$ and $\mathbf{c}$, as well as $\mathbf{y}, \mathbf{p}$.

The matrices $\mathbf{B}$ and $\mathbf{R}$ associated with the control can be decomposed analogously. After a suitable reordering of rows and columns, the matrix $\mathbf{B}$ can be written as

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{II}^1 & & \\ & \ddots & \\ & & \mathbf{B}_{II}^s \\ (\mathbb{I}_1^y)^T \mathbf{B}_{\Gamma I}^1 & \cdots & (\mathbb{I}_s^y)^T \mathbf{B}_{\Gamma I}^s \end{pmatrix}.$$

Note that due to our control discretization, there are not controls associated with the interface Γ. Consequently, there are no $\mathbf{B}_{I\Gamma}^i, \ldots, \mathbf{B}_{\Gamma\Gamma}^i$. The matrix $\mathbf{R}$ and the vector $\mathbf{u}$ can be decomposed analogously. Note that there is no $\mathbf{u}_\Gamma$.

We can now insert the domain decomposition structure of the matrices $\mathbf{A}, \mathbf{Q}, \mathbf{B}, \mathbf{R}$ into (4.2). After a symmetric permutation, (4.2) can be written as

$$\begin{pmatrix} \mathbf{K}_{II}^1 & & & (\mathbf{K}_{\Gamma I}^1)^T \mathbb{I}_1 \\ & \ddots & & \vdots \\ & & \mathbf{K}_{II}^s & (\mathbf{K}_{\Gamma I}^s)^T \mathbb{I}_s \\ \mathbb{I}_1^T \mathbf{K}_{\Gamma I}^1 & \cdots & \mathbb{I}_s^T \mathbf{K}_{\Gamma I}^s & \mathbf{K}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{x}_I^1 \\ \vdots \\ \mathbf{x}_I^d \\ \mathbf{x}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{g}_I^1 \\ \vdots \\ \mathbf{g}_I^d \\ \mathbf{g}_\Gamma \end{pmatrix}, \tag{4.5}$$

where

$$\mathbf{K}_{\Gamma\Gamma}^i = \begin{pmatrix} \mathbf{Q}_{\Gamma\Gamma}^i & (\mathbf{A}_{\Gamma\Gamma}^i)^T \\ \mathbf{A}_{\Gamma\Gamma}^i & \end{pmatrix}, i = 1, \ldots, s, \quad \mathbf{K}_{\Gamma\Gamma} = \sum_{i=1}^s \mathbb{I}_i^T \mathbf{K}_{\Gamma\Gamma}^i \mathbb{I}_i,$$

$$\mathbf{K}_{II}^i = \begin{pmatrix} \mathbf{Q}_{II}^i & \mathbf{0} & (\mathbf{A}_{II}^i)^T \\ \mathbf{0} & \alpha \mathbf{R}_{II}^i & (\mathbf{B}_{II}^i)^T \\ \mathbf{A}_{II}^i & \mathbf{B}_{II}^i & \end{pmatrix}, \quad \mathbf{K}_{\Gamma I}^i = \begin{pmatrix} \mathbf{Q}_{\Gamma I}^i & \mathbf{0} & (\mathbf{A}_{I\Gamma}^i)^T \\ \mathbf{A}_{\Gamma I}^i & \mathbf{B}_{\Gamma I}^i & \end{pmatrix}.$$

Furthermore,

$$\mathbf{x}_\Gamma = \begin{pmatrix} \mathbf{y}_\Gamma \\ \mathbf{p}_\Gamma \end{pmatrix}, \ \mathbf{g}_\Gamma = \begin{pmatrix} \mathbf{c}_\Gamma \\ \mathbf{b}_\Gamma \end{pmatrix}, \quad \mathbf{x}_I^i = \begin{pmatrix} \mathbf{y}_I^i \\ \mathbf{u}_I^i \\ \mathbf{p}_I^i \end{pmatrix}, \ \mathbf{g}_I^i = \begin{pmatrix} \mathbf{c}_I^i \\ \mathbf{d}_I^i \\ \mathbf{b}_I^i \end{pmatrix}.$$

Frequently, we use the compact notation

$$\begin{pmatrix} \mathbf{K}_{II} & \mathbf{K}_{\Gamma I}^T \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{x}_I \\ \mathbf{x}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{g}_I \\ \mathbf{g}_\Gamma \end{pmatrix}, \tag{4.6}$$

or even $\mathbf{K}\mathbf{x} = \mathbf{g}$ instead of (4.5).

Assuming that $\mathbf{K}_{II}$ is invertible (we will present conditions that guarantee the invertibility in Theorem 4.2 below), we can form the Schur complement system

$$\mathbf{S}\mathbf{x}_\Gamma = \mathbf{r} \tag{4.7}$$

corresponding to (4.5), where

$$\mathbf{S} = \mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{K}_{\Gamma I}^T \tag{4.8}$$

and

$$\mathbf{r} = \mathbf{g}_\Gamma - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{g}_I.$$

Due to the block structure of $\mathbf{K}_{\Gamma I}$ and $\mathbf{K}_{II}$, the Schur complement $\mathbf{S}$ can be written as a sum of subdomain Schur complements,

$$\mathbf{S} = \sum_{i=1}^{s} \mathbb{I}_i^T \mathbf{S}_i \mathbb{I}_i, \tag{4.9}$$

where

$$\mathbf{S}_i = \mathbf{K}_{\Gamma\Gamma}^i - \mathbf{K}_{\Gamma I}^i(\mathbf{K}_{II}^i)^{-1}(\mathbf{K}_{\Gamma I}^i)^T, \quad i = 1, \ldots, s. \tag{4.10}$$

Similarly,

$$\mathbf{r} = \sum_{i=1}^{s} \mathbb{I}_i^T \mathbf{r}_i,$$

where $\mathbf{r}_i = \mathbf{g}_\Gamma^i - \mathbf{K}_{\Gamma I}^i(\mathbf{K}_{II}^i)^{-1}\mathbf{g}_I^i$, $i = 1, \ldots, s$.

Observe that

$$\mathbf{S}_i = \mathbf{H}_i^T \mathbf{K}^i \mathbf{H}_i, \tag{4.11}$$

where

$$\mathbf{H}_i = \begin{pmatrix} -(\mathbf{K}_{II}^i)^{-1}\mathbf{K}_{I\Gamma}^i \\ \mathbf{I} \end{pmatrix} \tag{4.12}$$

and

$$\mathbf{K}^i = \begin{pmatrix} \mathbf{K}_{II}^i & (\mathbf{K}_{\Gamma I}^i)^T \\ \mathbf{K}_{\Gamma I}^i & \mathbf{K}_{\Gamma\Gamma}^i \end{pmatrix}.$$

As before (see note below equation (4.10)), the application of $\mathbb{I}_i$ and $\mathbb{I}_i^T$ eliminate zero rows and columns.

The matrix $\mathbf{H}_i$ defined in (4.12) is the matrix representation of the operator $\mathcal{H}_i^h$ defined in (3.17). The representation (4.11) corresponds to the representation (3.18b) of the subdomain Schur complement operator $\mathcal{S}_i^h$.

The matrix $\mathbf{K}^i$ plays an important role for the computation of the inverse of $\mathbf{S}_i$ (assuming it exists), which will be used in Section 4.2 to precondition $\mathbf{S}$. In fact, if $\mathbf{K}_{II}^i$ is invertible,

$$\mathbf{K}^i = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbb{I}_i\mathbf{K}_{\Gamma I}^i(\mathbf{K}_{II}^i)^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{K}_{II}^i & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_i \end{pmatrix} \begin{pmatrix} \mathbf{I} & (\mathbf{K}_{II}^i)^{-1}(\mathbf{K}_{\Gamma I}^i)^T\mathbb{I}_i^T \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \tag{4.13}$$

and $\mathbf{S}_i$ is invertible if and only if $\mathbf{K}^i$ is invertible. In this case,

$$\mathbf{S}_i^{-1}\mathbf{v} = (\mathbf{0} \quad \mathbf{I})\,(\mathbf{K}^i)^{-1}\begin{pmatrix} \mathbf{0} \\ \mathbf{I} \end{pmatrix}\mathbf{v} \tag{4.14}$$

(see, e.g., [33, p. 113]). The previous formula is the algebraic version of Theorem 3.4.

We conclude this subsection with a result concerning the invertibility of the submatrices $\mathbf{K}^i_{II}$, which is important for the computation of $\mathbf{S}_i$, and with the invertibility of the submatrices $\mathbf{K}^i$, which is important for the computation of $(\mathbf{S}_i)^{-1}$. We set

$$\mathbf{A}^i = \begin{pmatrix} \mathbf{A}^i_{II} & \mathbf{A}^i_{I\Gamma} \\ \mathbf{A}^i_{\Gamma I} & \mathbf{A}^i_{\Gamma\Gamma} \end{pmatrix}.$$

Matrices $\mathbf{Q}^i, \mathbf{R}^i$ are defined analogously.

Before we state our result on the invertibility of $\mathbf{K}^i_{II}$ and $\mathbf{K}^i$, we recall the following theorem, which is proven, e.g., in [14].

Theorem 4.1 *Let $\mathbf{A} \in \mathbb{R}^{m\times m}, \mathbf{B} \in \mathbb{R}^{m\times n}$ be arbitrary matrices and let $\mathbf{Q} \in \mathbb{R}^{m\times m}, \mathbf{R} \in \mathbb{R}^{n\times n}$ be symmetric. If*

$$\textit{range}(\mathbf{A} \mid \mathbf{B}) = \mathbb{R}^m \tag{4.15}$$

and if

$$\begin{pmatrix} \mathbf{z} \\ \mathbf{v} \end{pmatrix}^T \begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \alpha\mathbf{R} \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \mathbf{v} \end{pmatrix} > 0 \tag{4.16}$$

for all $\mathbf{z} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{Az} + \mathbf{Bv} = 0$ and $(\mathbf{z}^T, \mathbf{v}^T) \neq 0$, then

$$\begin{pmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{A}^T \\ \mathbf{0} & \alpha\mathbf{R} & \mathbf{B}^T \\ \mathbf{A} & \mathbf{B} & \mathbf{0} \end{pmatrix}$$

has $m + n$ positive eigenvalues and m negative eigenvalues.

Theorem 4.2 *i. The matrices $\mathbf{Q}^i_{II}, \mathbf{R}^i_{II}$ are symmetric positive definite. If* (2.4a), (2.4b), (2.4d) *hold, and if the stabilization parameter τ_e is sufficiently small, the matrix $\mathbf{A}^i_{II}$ obeys $\mathbf{v}^T\mathbf{A}^i_{II}\mathbf{v} > 0$ for all $\mathbf{v} \neq 0$ and $\mathbf{K}^i_{II}$ is invertible.*

ii. The matrices $\mathbf{Q}^i, \mathbf{R}^i$ are symmetric positive definite. If (2.4a)–(2.4c) *hold and if the stabilization parameter τ_e is sufficiently small, the matrix $\mathbf{A}^i$ obeys $\mathbf{v}^T\mathbf{A}^i\mathbf{v} > 0$ for all $\mathbf{v} \neq 0$ and $\mathbf{K}^i$ is invertible.*

iii. If (2.4a), (2.4b), (2.4d) *hold, if the relative interior of $\partial\Omega_i \cap \partial\Omega_D$ is nonempty, and if the stabilization parameter τ_e is sufficiently small, the matrix $\mathbf{A}^i$ obeys $\mathbf{v}^T\mathbf{A}^i\mathbf{v} > 0$ for all $\mathbf{v} \neq 0$ and $\mathbf{K}^i$ is invertible.*

Proof: i. Using the definitions (3.8) and (3.9) and the assumptions (2.4a), (2.4b), (2.4d) we have

$$\begin{aligned} a_{i,h}(v_h, v_h) &= \int_{\Omega_i} \epsilon\nabla v_h(x)\cdot\nabla v_h(x) + (r(x) - \tfrac{1}{2}\nabla\cdot\mathbf{a}(x))v_h^2(x)dx \\ &\quad + \tfrac{1}{2}\int_{\partial\Omega_i\cap\partial\Omega_N} \mathbf{a}(x)\cdot\mathbf{n}_i\, v_h^2(x)dx, \\ &\quad + \sum_{T_e\in\overline{\Omega_i}} \tau_e\langle -\epsilon\Delta v_h + \mathbf{a}\cdot\nabla v_h + rv_h, \mathbf{a}\cdot\nabla v_h\rangle_{T_e} \\ &\geq \epsilon|\nabla v_h|_{1,\Omega_i} + r_0\|v_h\|_{0,\Omega_i} \\ &\quad + \sum_{T_e\in\overline{\Omega_i}} \tau_e\langle -\epsilon\Delta v_h + \mathbf{a}\cdot\nabla v_h + rv_h, \mathbf{a}\cdot\nabla v_h\rangle_{T_e} \end{aligned}$$

for all $v_h \in Y_{i,0}^h$. (Note that for $v_h \in Y_{i,0}^h$ we have $a_{i,h}(v_h, v_h) = \widetilde{a}_{i,h}(v_h, v_h)$, i.e., the modification of the local bilinear from (3.8) is not important here.) Standard SUPG estimates (cf. [25, p. 378] or [30, L. 3.28,p. 231]) show that

$$a_{i,h}(v_h, v_h) \geq \frac{\epsilon}{2}|\nabla v_h|_{1,\Omega_i} + \frac{r_0}{2}\|v_h\|_{0,\Omega_i} + \frac{1}{2}\sum_{T_e \in \overline{\Omega_i}} \tau_e \|\mathbf{a} \cdot \nabla v_h\|_{0,T_e}$$

for all $v_h \in Y_{i,0}^h$. By a Poincaré inequality, we have $\|v_h\|_{0,\Omega_i} \leq c|\nabla v_h|_{1,\Omega_i}$. Hence,

$$\mathbf{v}^T \mathbf{A}_{II}^i \mathbf{v} = a_{i,h}(v_h, v_h) \geq \frac{\epsilon}{2}|\nabla v_h|_{1,\Omega_i} + \frac{r_0}{2}\|v_h\|_{0,\Omega_i} + \frac{1}{2}\sum_{T_e \in \overline{\Omega_i}} \tau_e \|\mathbf{a} \cdot \nabla v_h\|_{0,T_e} > 0$$

for all $\mathbf{v} \neq 0$. In particular $\mathbf{A}_{II}^i$ is invertible and (4.15) with $\mathbf{A}$, $\mathbf{B}$, m replaced by $\mathbf{A}_{II}^i$, $\mathbf{B}_{II}^i$, m_I^i, respectively, is valid. Moreover, the matrices $\mathbf{Q}_{II}^i \in \mathbb{R}^{m_I^i \times m_I^i}$ $\mathbf{R}_{II}^i \in \mathbb{R}^{n^i \times n^i}$ are subdomain mass matrices, which implies their symmetric positive definiteness. Hence (4.16) with $\mathbf{Q}, \mathbf{R}$ replaced by $\mathbf{Q}_{II}^i, \mathbf{R}_{II}^i$ is valid for all $\mathbf{z} \in \mathbb{R}^{m_I^i}, \mathbf{v} \in \mathbb{R}^{n^i}$ with $(\mathbf{z}^T, \mathbf{v}^T) \neq 0$. The result now follows from Theorem 4.1.

ii. We proceed as in the first part to show that

$$a_{i,h}(v_h, v_h) \geq \frac{\epsilon}{2}|\nabla v_h|_{1,\Omega_i} + \frac{r_0}{2}\|v_h\|_{0,\Omega_i} + \frac{1}{2}\sum_{T_e \in \overline{\Omega_i}} \tau_e \|\mathbf{a} \cdot \nabla v_h\|_{0,T_e} \quad \forall v_h \in Y_i^h.$$

Hence, $\mathbf{v}^T \mathbf{A}^i \mathbf{v} > 0$ for all $\mathbf{v} \neq 0$. We can now proceed as in part i. to prove the invertibility of $\mathbf{K}^i$.

iii. If the relative interior of $\partial\Omega_i \cap \partial\Omega_D$ is nonempty, then due to a Poincaré inequality there exists a constant $c > 0$ such that $\|v_h\|_{0,\Omega_i} \leq c|\nabla v_h|_{1,\Omega_i}$ for all $v_h \in Y_i^h$ and we can admit $r_0 = 0$ in part ii. □

Remark 4.3 *i. Examination of the proof of Theorem* 4.2 *reveals the importance of the modification* (3.8) *of the local bilinear form to guarantee* $a_{i,h}(v_h, v_h) > 0$ *for all* $v_h \in Y_i^h, v_h \neq 0$, *i.e.,* $\mathbf{v}^T \mathbf{A}^i \mathbf{v} > 0$ *for all* $\mathbf{v} \neq 0$.

ii. Just to guarantee the invertibility of $\mathbf{K}^i$, *the conditions in Theorem* 4.2*ii, iii. may be too strong.*

For our model problem with distributed control, $\mathbf{B}^i \in \mathbb{R}^{m^i \times n^i}$, *with* $n^i > m^i$, *is related to the mass matrix and satisfies rank*$(\mathbf{B}^i) = \mathbb{R}^{m^i}$. *Hence,* (4.15) *is satisfied. (The invertibility of* $\mathbf{A}^i$ *is not needed.) Moreover,* $\mathbf{Q}^i, \mathbf{R}^i$ *are subdomain mass matrices, and, hence,*

$$\begin{pmatrix} \mathbf{z} \\ \mathbf{v} \end{pmatrix}^T \begin{pmatrix} \mathbf{Q}^i & \mathbf{0} \\ \mathbf{0} & \alpha \mathbf{R}^i \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \mathbf{v} \end{pmatrix} > 0 \quad \textit{for all } \mathbf{z} \in \mathbb{R}^{m^i}, \mathbf{v} \in \mathbb{R}^{n^i} \textit{ with } (\mathbf{z}^T, \mathbf{v}^T) \neq 0.$$

This means that for our model problem with distributed control, the invertibility of $\mathbf{A}^i$ *is not needed to ensure the invertibility of* $\mathbf{K}^i$*! In particular,* $\mathbf{K}^i$ *is also invertible if we use the local bilinear form* (3.7) *instead of* (3.8).

4.2 The Robin-Robin Preconditioners

It is now relatively easy to generalize the Robin-Robin preconditioner used in the context of advection dominated elliptic PDEs [4] to the optimal control context.

Let $\mathbf{D}_i^y$ be the diagonal matrix, whose entries are computed as follows. If the node x_k satisfies $x_k \in \Gamma_i$, then $(\mathbf{D}_i^y)_{kk}^{-1}$ is the number of subdomains that share node x_k. Note that $\sum_i \mathbf{D}_i^y = \mathbf{I}$. Furthermore, let $\tilde{\mathbf{D}}_i^p = \tilde{\mathbf{D}}_i^y$ and

$$\mathbf{D}_i = \begin{pmatrix} \mathbf{D}_i^y & \\ & \mathbf{D}_i^p \end{pmatrix}.$$

By Theorem 4.2 i. $\mathbf{S}_i$, $i = 1, \ldots, s$, is well defined. The one-level Robin-Robin preconditioner is given by

$$\mathbf{P} = \sum_i \mathbf{D}_i \mathbb{I}_i^T \mathbf{S}_i^{-1} \mathbb{I}_i \mathbf{D}_i. \tag{4.17}$$

In principle it is possible to incorporate a coarse space, but this has not yet been been explored in the optimal control context.

4.3 Implementation

Instead of working on the preconditioned Schur complement system

$$\mathbf{P}\mathbf{S}\mathbf{x}_\Gamma = \mathbf{P}(\mathbf{g}_\Gamma - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{g}_I) = \mathbf{P}\mathbf{r}. \tag{4.18}$$

we work on the preconditioned full system. It is easy to verify that

$$\begin{pmatrix} \mathbf{K}_{II} & \mathbf{K}_{\Gamma I}^T \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{K}_{II} & 0 \\ \mathbf{K}_{\Gamma I} & \mathbf{P}^{-1} \end{pmatrix}}_{=(\mathbf{P}_l^K)^{-1}} \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{P}\mathbf{S} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{I} & \mathbf{K}_{II}^{-1}\mathbf{K}_{\Gamma I}^T \\ 0 & \mathbf{I} \end{pmatrix}}_{=(\mathbf{P}_r^K)^{-1}}. \tag{4.19}$$

We will look at the preconditioned system

$$\mathbf{P}_l^K \mathbf{K} \mathbf{P}_r^K \widehat{\mathbf{x}} = \mathbf{P}_l^K \mathbf{g}, \tag{4.20}$$

where $\widehat{\mathbf{x}} = (\mathbf{P}_r^K)^{-1}\mathbf{x}$, and at the preconditioned Schur complement system (4.18). Consider an initial iterate

$$\mathbf{x}^0 = \begin{pmatrix} \mathbf{x}_I^0 \\ \mathbf{x}_\Gamma^0 \end{pmatrix}, \tag{4.21}$$

with

$$\mathbf{x}_I^0 = \mathbf{K}_{II}^{-1}(\mathbf{g}_I - \mathbf{K}_{\Gamma I}^T \mathbf{x}_\Gamma^0) \tag{4.22}$$

and set $\widehat{\mathbf{x}}^0 = (\mathbf{P}_r^K)^{-1}\mathbf{x}^0$. The corresponding preconditioned residual satisfies

$$\widehat{\mathbf{r}}^0 = \mathbf{P}_l^K(\mathbf{g} - \mathbf{K}\mathbf{P}_r^K)\widehat{\mathbf{x}}^0 = \begin{pmatrix} 0 \\ \mathbf{P}(\mathbf{g}_\Gamma - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{g}_I - \mathbf{S}\mathbf{x}_\Gamma^0) \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{\mathbf{r}}_\Gamma^0 \end{pmatrix}. \tag{4.23}$$

We see that the second component of the initial residual $\widehat{\mathbf{r}}^0$ of the preconditioned system (4.20) is the initial residual $\widehat{\mathbf{r}}^0_\Gamma = \mathbf{P}(\mathbf{g}_\Gamma - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{g}_I - \mathbf{S}\mathbf{x}^0_\Gamma)$ of the preconditioned Schur complement system (4.18).

Recall that for a matrix A and a vector v, the Krylov subspace is defined by $\mathcal{K}_k(A, v) = \text{span}\{v, Av, \ldots, A^{k-1}v\}$. Using the fact that the first component of $\widehat{\mathbf{r}}^0$ is zero and that $\mathbf{P}_l^K\mathbf{K}\mathbf{P}_r^K$ is a block diagonal matrix, we immediately obtain the following relation between the Krylov subspaces of the preconditioned system (4.20) and the preconditioned Schur complement system (4.18):

$$\mathcal{K}_k(\mathbf{P}_l^K\mathbf{K}\mathbf{P}_r^K, \widehat{\mathbf{r}}^0) = \{0\} \times \mathcal{K}_k(\mathbf{P}\mathbf{S}, \widehat{\mathbf{r}}^0_\Gamma) \quad \forall\, k. \tag{4.24}$$

This relationship allows one to establish relationships between Krylov subspace methods applied to the preconditioned Schur compement system (4.18) and the preconditioned full system (4.20), provided that the initial iterates satisfy (4.22). For the symmetric positive definite case see [24]. If the application of $\mathbf{K}_{II}^{-1}$ is exact, there is no difference between the solution of preconditioned Schur complement system (4.18) and the preconditioned full system (4.20). However, the latter provides advantages if the application of $\mathbf{K}_{II}^{-1}$ is performed inexactly using iterative methods [24, 18]. In our numerical examples, we solve systems of the form $\mathbf{K}_{II}^i\mathbf{v}_I^i = \mathbf{r}_I^i$ and $\mathbf{K}^i\mathbf{v}^i = \mathbf{r}^i$ (the latter arising in the application of our preconditioner, cf. (4.14)) exactly (up to floating point arithmetic) using UMFPACK 4.3 [11]. Still, we work with the the preconditioned full system (4.20) to allow the incorporation of iterative solvers in the future.

In our numerical experiments reported on in the next section, we use GMRES [31] and sQMR [12, 13] applied to

$$\mathbf{P}_r^K\mathbf{P}_l^K\mathbf{K}\mathbf{x} = \mathbf{P}_r^K\mathbf{P}_l^K\mathbf{g}. \tag{4.25}$$

We have observed that the number of GMRES [sQMR] iterations applied to (4.20) is close to the number of GMRES [sQMR] iterations applied to (4.25). In both cases GMRES [sQMR] was stopped if the respective preconditioned residual was reduced by a factor of 10^{-9}. However, we also observed that the error between the solution computed using GMRES and the exact solution $\mathbf{K}^{-1}\mathbf{g}$ was for small diffusion ϵ significantly smaller when left preconditioning (4.25) was used instead of split preconditioning (4.20). This is not surprising, since the GMRES iteration is stopped when the preconditioned residual $\|\mathbf{P}_r^K\mathbf{P}_l^K\mathbf{K}\mathbf{x} - \mathbf{P}_r^K\mathbf{P}_l^K\mathbf{g}\|$ or $\|\mathbf{P}_l^K\mathbf{K}\mathbf{x} - \mathbf{P}_l^K\mathbf{g}\|$, respectively, is small and the matrix $\mathbf{P}_r^K\mathbf{P}_l^K\mathbf{K}$ is expected to have a smaller condition number than $\mathbf{P}_l^K\mathbf{K}$. The error between the solution computed using sQMR applied to (4.25) and the exact solution $\mathbf{K}^{-1}\mathbf{g}$ was observed to be also smaller than the error between the solution computed using sQMR applied to (4.20) and the exact solution $\mathbf{K}^{-1}\mathbf{g}$, but the differences were much smaller than those observed for GMRES.

5 Numerical Results

In this section we illustrate the performance of our optimization-level domain decomposition method for several advection dominated optimal control problems with distributed controls or with boundary controls. Additional numerical examples may be found in [6].

To explore the importance of the modification (3.8) of the local bilinear form, we run experiments with and without this modification. If we use the modified local bilinear form (3.8), then we refer to the resulting preconditioner as a Robin–Robin (R–R) preconditioner. This name is motivated by the Robin transmission conditions (3.15) for the state (and the adjoint) and the Robin boundary conditions for the state (and the

adjoint) in the subproblem (3.27) for the inversion of $\mathcal{S}_i$. If $a_i(y_h, \phi_h) = \widetilde{a}_i(y_h, \phi_h)$, i.e., no modification of the local bilinear form is applied, then we refer to the resulting preconditioner as a Neumann–Neumann (N–N) preconditioner. This name is motivated by the Neumann transmission conditions (3.16) for the state and the Neumann boundary conditions for the state in the subproblem (3.29) for the inversion of $\mathcal{S}_i$.

5.1 Distributed Control

Example 1: Influence of different velocity fields. This example is derived from Example 4.1 in [4]. We use $\Omega = (0,1) \times (0,0.2)$, $\partial\Omega_D = \partial\Omega$, $r = 1$, $f = 0$, and one of the following four advections $\mathbf{a}(x) = \mathbf{e}_1$, $\mathbf{a}(x) = \mathbf{e}_2$, $\mathbf{a}(x) = (1/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2)$, or $\mathbf{a}(x) = 2\pi((x_1 - 0.5)\mathbf{e}_2 + (x_2 - 0.1)\mathbf{e}_1)$. These are referred to as 'normal', 'parallel', 'oblique', and 'rotating', respectively. We generate $\hat{y}$ as the solution of

$$-\epsilon\Delta\hat{y}(x) + \mathbf{a}(x)\cdot\nabla\hat{y}(x) + \hat{y}(x) = 5e^{-\frac{(x_1-0.2)^2+(x_2-0.1)^2}{2\cdot 0.1^2}} + 5e^{-\frac{(x_1-0.8)^2+(x_2-0.1)^2}{2\cdot 0.1^2}}, \quad x \in \Omega, \tag{5.1a}$$

$$\hat{y}(x) = 0, \quad x \in \partial\Omega. \tag{5.1b}$$

We decompose Ω into 5 subdomains of size $(0,0.2) \times (0,0.2)$. Each subdomain is triangulated by dividing each axis into 30 subintervals and subsequently subdividing the resulting rectangles into two triangles. The problems are solved by a preconditioned sQMR algorithm where the stopping criterion is to reduce the initial residual by a factor of 10^{-9}. We use either Robin–Robin (R–R) or Neumann–Neumann (N–N) preconditioning.

Unlike in the PDE-only case in [4], the unpreconditioned sQMR (the same is true for GMRES) fails to reduce the initial residual to the specified tolerance within 1000 iterations for all experiments outlined below. Therefore, we do not give any further numerical results in absence of preconditioning. In Tables 5.1 and 5.2 we report the number of preconditioned sQMR iterations for the values $\alpha = 10^{-4}$ and $\alpha = 1$ of the regularization parameter, respectively.

We recall (cf. Remark 4.3) that for the distributed control case the invertibility of $\mathbf{A}^i$, i.e., the modification (3.8) of the bilinear form is not needed to ensure invertibility of $\mathbf{K}^i$ and, hence $\mathbf{S}_i$. Thus the application of the Neumann–Neumann (N–N) preconditioner is well–posed for the distributed control case.

ϵ	Prec. \ Velocity	Normal	Parallel	Oblique	Rotating
0.001	R-R	12	3	13	9
	N-N	21	3	18	13
1	R-R	4	4	4	4
	N-N	4	4	4	4

Table 5.1: sQMR iterations for different velocity fields, $\alpha = 10^{-4}$.

Tables 5.1 and 5.2 show that for large ϵ, both Robin–Robin and Neumann–Neumann preconditioners perform equally well, with all sQMR runs finishing in 4 iterations. This is in agreement with the PDE-only case reported in [4, Table 1]. When the velocity is parallel to subdomain interfaces, then $a_i(y_h, \phi_h) = \widetilde{a}_i(y_h, \phi_h)$ and the Robin–Robin and the Neumann-Neumann are identical. The Robin–Robin preconditioner adapts nicely to small ϵ for all velocities. The performance of the Neumann–Neumann preconditioner deteriorates with decreasing ϵ, but this deterioration is not nearly as pronounced as in the PDE-only case

ϵ	Prec. \ Velocity	Normal	Parallel	Oblique	Rotating
0.001	R-R	12	3	4	6
	N-N	53	3	30	14
1	R-R	4	4	4	4
	N-N	4	4	4	4

Table 5.2: sQMR iterations for different velocity fields, $\alpha = 1$.

in [4, Table 1]. Finally, we observe that the size of the regularization parameter α seems to affect the performance of both preconditioners only moderately.

Example 2: Influence of the number of subdomains, grid sizes, and regularization. The purpose of this example is to assess the sensitivity of the Robin–Robin and Neumann–Neumann preconditioners to increases in the number of subdomains and grid points.

We use $\Omega = (0,1) \times (0,1)$, $\partial\Omega_D = \partial\Omega$, $\epsilon = 0.001$, $\mathbf{a}(x) = 3\mathbf{e}_1$ $r = 1$, and $f = 0$. We generate $\hat{y}$ as in (5.1) but with right hand side replaced by

$$5e^{-\frac{(x_1-0.2)^2+(x_2-0.1)^2}{2\cdot 0.1^2}} + 5e^{-\frac{(x_1-0.8)^2+(x_2-0.9)^2}{2\cdot 0.1^2}}.$$

For the first experiment we use a fixed uniform grid of size 128×128 (note that each square in the mesh is divided into two triangles). The grid is partitioned in various ways. First, we use 4, 8, and 16 vertical rectangular strips of equal size (yielding subdomain sizes of 32×128, 16×128, and 8×128, respectively). Second, we partition the grid into 2×2, 4×4, 8×8, and 16×16 square subdomains (with sudomain sizes of 64×64, 32×32, 16×16, and 8×8 respectively). Finally, the grid is subdivided into 16 horizontal rectangular strips of equal size (yielding a subdomain size of 128×8). The results are presented in Table 5.3.

Reg.	Prec. \ Part.	4×1	8×1	16×1	2×2	4×4	8×8	16×16	1×16
$\alpha = 10^{-4}$	R-R	12	12	14	13	15	17	21	3
	N-N	39	39	38	35	44	46	49	3
$\alpha = 1$	R-R	9	19	38	7	14	24	47	3
	N-N	130	361	> 500	87	172	452	> 500	3

Table 5.3: sQMR iterations for varying numbers of subdomains, $\epsilon = 0.001$.

Table 5.3 shows that for large α, the number of sQMR iterations roughly doubles as the number of subdomains in the direction of the velocity field is doubled, for both preconditioners. This is also observed in [4, Table 2]. The Robin–Robin preconditioner performs better than the Neumann-Neumann preconditioner. For large α the performance differences are as pronounced as in the PDE-only case reported in [4].

For small α, the number of sQMR iterations does not increase significantly as the number of subdomains is increased (regardless of the position of subdomain interfaces). This is a surprising and not yet understood result, which unfortunately does not hold true for most other problem setups with complex velocity fields

(see [6, Sec. 5.2]). The Neumann–Neumann preconditioner performs much better here than in the case of large α.

For partitions in which all subdomain interfaces are parallel to the velocity field, i.e., 1×4, 1×8, and 1×16 partitions, the number of sQMR iterations is not affected at all by the number of subdomains or the size of the regularization parameter. Both Robin–Robin and Neumann–Neumann preconditioned sQMR runs complete all tests in only 3 iterations (the 1×4 and 1×8 results are not tabulated).

The second experiment examines the influence of the number of grid points. The problem is set up as in the first experiment, except that here we fix two particular subdomain partitions, and vary the grid size. We use either an 8×1 rectangular subdomain partition or a 4×4 square subdomain partition, on uniform grids of sizes 32×32, 64×64, and 128×128 (again, each mesh square is split into two triangular elements). The results are presented in Table 5.4.

		8×1 Partition			4×4 Partition		
Reg.	Prec. \ Full Grid	32×32	64×64	128×128	32×32	64×64	128×128
$\alpha = 10^{-4}$	R-R	15	12	12	16	16	15
	N-N	21	27	39	28	33	44
$\alpha = 1$	R-R	21	19	19	16	15	14
	N-N	307	368	361	143	156	172

Table 5.4: sQMR iterations for varying numbers of grid points, $\epsilon = 0.001$.

They indicate that the convergence of the sQMR algorithm with the Robin–Robin preconditioner is not affected by the grid size. This agrees with the results stated in [4, Table 5]. On the other hand, the performance of the Neumann-Neumann preconditioned algorithm deteriorates slightly as the number of grid points is increased. The size of the regularization parameter α does not affect the performance of the Robin–Robin preconditioner. In contrast, for large α the Neumann-Neumann preconditioner performs extremely poorly for all grid sizes.

5.2 Robin Boundary Control

The domain decomposition method described in the previous section for optimal control problems with distributed control can be extended to problems with boundary control using the ideas in [21]. We report on some numerical results for the example problem

$$\text{minimize } \frac{1}{2}\int_{\Omega}(y(x)-\widehat{y}(x))^2dx + \frac{\alpha}{2}\int_{\partial\Omega_c} u^2(x)dx \tag{5.2a}$$

subject to

$$-\epsilon\Delta y(x) + \mathbf{a}(x)\cdot\nabla y(x) + r(x)y(x) = f(x), \qquad x \in \Omega, \tag{5.2b}$$

$$y(x) = 0, \qquad x \in \partial\Omega_D, \tag{5.2c}$$

$$\epsilon\frac{\partial}{\partial\mathbf{n}}y(x) + \delta y(x) = \delta u(x), \qquad x \in \partial\Omega_c \tag{5.2d}$$

where $r = 1$, $f = 0$, and $\delta = 10^3$. The Robin boundary condition (5.2d) can be viewed as a penalized Dirichlet condition [5, 22].

Example 1: Influence of different velocity fields. The problems data are the same as those in Example 1 in Section 5.1. We examine the performance of the Robin–Robin and Neumann-Neumann preconditioners with respect to various velocity fields, on the rectangular domain $\Omega = (0, 1) \times (0, 0.2)$ with five square subdomains. We choose $\partial\Omega_c = \partial\Omega$, i.e. $\partial\Omega_D = \emptyset$. In Tables 5.5 and 5.6 we report the number of preconditioned sQMR iterations for the values $\alpha = 10^{-4}$ and $\alpha = 1$, respectively.

ϵ	Prec. \ Velocity	Normal	Parallel	Oblique	Rotating
0.001	R-R	16	3	9	9
	N-N	129	3	36	17
1	R-R	4	4	4	4
	N-N	4	4	4	4

Table 5.5: sQMR iterations for different velocity fields, $\alpha = 10^{-4}$.

ϵ	Prec. \ Velocity	Normal	Parallel	Oblique	Rotating
0.001	R-R	7	3	3	6
	N-N	73	3	29	15
1	R-R	4	4	4	4
	N-N	4	4	4	4

Table 5.6: sQMR iterations for different velocity fields, $\alpha = 1$.

The obtained results are similar to those in Section 5.1, with one important difference. For small ϵ, the Neumann-Neumann preconditioner performs significantly worse when compared to the distributed control case. This behavior can be explained by re-examining Remark 4.3. The boundary control problem lacks the property rank$(\mathbf{B}^i) = \mathbb{R}^{m^i}$. Therefore, the invertibility of $\mathbf{A}^i$ is now needed to ensure the invertibility of $\mathbf{K}^i$. Within the Neumann-Neumann preconditioner (i.e., no modification of the local bilinear form a_i), we have observed severely ill-conditioned $\mathbf{K}^i$'s in some subdomains (with estimated condition numbers of 10^6).

Example 2: Influence of the number of subdomains, grid sizes, and regularization. The second experiment assesses the sensitivity of the Robin–Robin and Neumann–Neumann preconditioners to increases in the number of subdomains. We use the same setup as in Example 2 in Section 5.1, i.e. the square domain $\Omega = (0, 1) \times (0, 1)$ with various partitioning schemes. As before, the velocity is $\mathbf{a}(x) = 3\mathbf{e}_1$ and $\epsilon = 0.001$. The results are presented in Table 5.7.

There are several major differences compared to the distributed control case. For the Robin–Robin preconditioner, the number of sQMR iterations roughly doubles as the number of subdomains in the direction of the velocity field doubles, regardless of the size of the regularization parameter α (i.e. small α does not yield partition independence). The failure of the Neumann-Neumann preconditioning scheme is evident. The preconditioned sQMR algorithm fails to achieve the desired relative residual within 500 iterations for six test cases. This result reinforces our conjecture from the previous experiment. When the regularization

Reg.	Prec. \ Part.	4×1	8×1	16×1	2×2	4×4	8×8	16×16	1×16
$\alpha = 10^{-4}$	R-R	17	37	78	9	20	42	82	4
	N-N	> 500	> 500	> 500	151	> 500	> 500	> 500	4
$\alpha = 1$	R-R	7	14	28	5	11	19	36	3
	N-N	142	218	340	83	163	260	420	3

Table 5.7: sQMR iterations for varying numbers of subdomains, $\epsilon = 0.001$.

parameter is increased from $\alpha = 10^{-4}$ to $\alpha = 1$, the number of Robin–Robin preconditioned sQMR iterations is roughly reduced by a factor of two for all test cases. This result is more intuitive than the one in the distributed control example.

The third experiment examines the influence of the number of grid points. The problem is set up as in Example 2 in Section 5.1, where we fix two particular subdomain partitions (8×1 and 4×4), and vary the grid size. The results are presented in Table 5.8.

		8×1 Partition			4×4 Partition		
Reg.	Prec. \ Full Grid	32×32	64×64	128×128	32×32	64×64	128×128
$\alpha = 10^{-4}$	R-R	38	36	37	21	20	20
	N-N	> 500	> 500	> 500	> 500	> 500	> 500
$\alpha = 1$	R-R	14	14	14	12	12	11
	N-N	224	219	220	147	167	158

Table 5.8: sQMR iterations for varying numbers of grid points, $\epsilon = 0.001$.

The results indicate that the convergence of the sQMR algorithm with the Robin–Robin preconditioner is mesh independent. This agrees with the observations made in the distributed control case. It is difficult to draw any conclusions about the performance of the Neumann-Neumann preconditioner as a function of the increasing number of grid points, since it performs quite poorly even for large α, and entirely fails to reach the desired relative residual for small α.

6 Conclusions

We have introduced an optimization-level domain decomposition preconditioner for advection dominated linear-quadratic elliptic optimal control problems, which extends the work of [4, 3] to the optimization context.

The tasks required for the application of the domain decomposition preconditioner are closely related to what is required for the solution of the global optimal control problem. This allows code reuse and enables optimization-level parallelization of existing solvers for advection dominated linear-quadratic elliptic optimal control problems.

Numerical experiments have shown that the preconditioner is fairly insensitive to the velocity, the viscosity and the control regularization parameter. For distributed control and Robin boundary control test

problems the preconditioner deteriorates only slowly as the number of subdomains is increased.

Unfortunately, a theoretical explanation for the performance of the preconditioner is not yet available. Theoretical investigations, the application of the preconditioner to other problems, in particular 3D problems, and the design and incorporation of coarse spaces into the preconditioner are part of future work.

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